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A modified Chan–Vese model and its theoretical proof ☆

Ling Pi [a,*], Yaxin Peng [b], Chunli Shen [c], Fang Li [c]

[a] *Department of Mathematics, Shanghai Jiao Tong University, Shanghai, China*
[b] *Department of Mathematics, Shanghai University, Shanghai, China*
[c] *Department of Mathematics, East China Normal University, Shanghai, China*

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ABSTRACT

We present a modified Chan–Vese functional and give its theoretical proof. By using the geometric heat flow method to all the Euler–Lagrange equations, a system of evolution equations in level set formulation is derived. We study the existence of solution to this system by Schauder fixed point theorem and the implicit function theorem in Banach space. This variational formulation can detect interior and exterior boundaries of desired object(s) in color images.

1. Introduction

In this paper we propose a model for image segmentation. Our aim is to segment desired object(s) from a color image. The proposed model incorporates a modified stopping function with the traditional Chan–Vese model [1,2]. This stopping function is based on the color information of the desired object(s).

The main contributions of this paper are in two aspects: application and theoretical aspects. For the application aspect, we focus on the segmentation of desired object(s) in color images, and mainly make use of the red, green and blue (RGB) information of the desired object(s). Using the color information and the derived gradient information, we propose a variational formulation that can detect the interior and exterior boundaries of the desired object(s). The color information is used to construct a discrimination function that determines whether a pixel belongs to the desired object(s) or not [3]. The discrimination function is included in the energy functional and the corresponding evolution equation. With this discrimination function, the evolving curve will stop near the desired objects. We calculate the first variation to the arc length energy functional, and then, we apply the geometric heat flow method and the level set method to all the Euler–Lagrange equations [4]. A system of evolution equations is derived. For the theoretical aspect, the existence of the above system of equations is proved by the viscosity solution theory [5], Schauder fixed point theorem and the implicit function theorem in Banach space [6,7].

This paper is organized as follows. In Section 2, we describe our variational formulation based on color information and derive the system of evolution equations. Section 3 discusses the existence of the solution. Our numerical algorithm and experimental results are given in Section 4. Section 5 concludes the paper.

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* Corresponding author.
E-mail addresses: piling@sjtu.edu.cn (L. Pi), juliettepeng@hotmail.com (Y. Peng), chunlishen@126.com (C. Shen), lifangswnu@126.com (F. Li).

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2. Modified Chan–Vese model

The geodesic active contour methods are very important in image segmentation [8,9]. However, they can only detect the exterior boundaries during the shrinkage. The Chan–Vese model is well known to segment all objects in a given image based on the Mumford–Shah model [10]. Kimmel presented a unified active contours framework based on these works [11]. The energy functional combines the geodesic active contour model, the Chan–Vese model, and a term called the alignment. The performances of Kimmel's method have been shown to be better than traditional geodesic active contours and the Chan–Vese model when directional information about the edge location is provided.

The above methods can detect all the objects from an image. Now, we present a modified Chan–Vese model to only detect the desired object(s) using the color information. Let $\vec{I} = (I_1, I_2, I_3) = (R, G, B)$ be defined on Ω: $\Omega \subset \mathbf{R}^2$. Let $\vec{C}$ be the evolving closed curve in the plane. The modified energy functional is

$$E(\vec{c}^+, \vec{c}^-, \vec{C}) = \mu \int_0^1 g(\vec{C}(p))|\vec{C}'_p|\,\mathrm{d}p + \nu \int_\omega g(\vec{C}(p))\,\mathrm{d}A + \frac{1}{3}\int_\omega \beta(x)\sum_{i=1}^3 \lambda_i^+ |I_i(x) - c_i^+|^2\,\mathrm{d}x$$
$$+ \frac{1}{3}\int_{\Omega\setminus\bar{\omega}} \beta(x)\sum_{i=1}^3 \lambda_i^- |I_i(x) - c_i^-|^2\,\mathrm{d}x,$$

where ω is the inside of the curve $\vec{C}$, $\vec{c}^+ = (c_1^+, c_2^+, c_3^+)$, $\vec{c}^- = (c_1^-, c_2^-, c_3^-)$, c_i^+ and c_i^- $(i = 1, 2, 3)$ are constants depending on $\vec{C}$, $\lambda_i^+, \lambda_i^- > 0$ are parameters for each channel, $\mu \geqslant 0$, ν are parameters, $g(x) = \frac{1}{1+(G_\sigma * \gamma(x))\cdot \Lambda^2}$ is the modified stopping function, where Λ is the largest eigenvalue of the metric tensor in the $\{x, y, R, G, B\}$ space [12], and $\gamma(x)$ is a discrimination function. Here $\beta(x) = G_\sigma * \gamma(x)$.

$\gamma(x)$ provides a probabilistic criterion for deciding whether a pixel is in the desired object(s) or not. We analyze n sample pixels chosen from the desired object(s) using the Principal Components Analysis (PCA) and interval estimation [13]. The PCA is used to transform the original coordinate system into a new coordinate, so that the RGB value $I(x)$ of each pixel of the color image is projected to the first principal component axis. Then we get a new value $\bar{I}(x)$. The interval estimation is used to define an interval $[a, b]$ that almost covers all samples for this principal component. Comparing the value with the interval, the pixel is regarded as a pixel in the desired object(s) in the sense of probability if and only if its value is within the interval. The discrimination function $\gamma(x)$ is:

$$\gamma(x) = \begin{cases} 1, & a \leqslant \bar{I}(x) \leqslant b; \\ 0, & \text{others}. \end{cases}$$

For the details of the discrimination function, please see [3].

In what follows, for simplicity, the constant $\frac{1}{3}$ is combined into the parameters λ_i^+ and λ_i^-.

Minimizing $E(\vec{c}^+, \vec{c}^-, \vec{C})$ with respect to the constants c_i^+, c_i^- $(i = 1, 2, 3)$ and the curve $\vec{C}$, it yields the associated Euler–Lagrange equations for c_i^+, c_i^- $(i = 1, 2, 3)$ and $\vec{C}$ respectively. The level set function $u(t, x)$ is used to get the implicit representation of the evolving curve $\vec{C}$ with the family parameter t [14,15]:

$$\vec{C} = \{x \in \mathbf{R}^2\colon\ u(t, x) = 0\}.$$

The equation can be solved using a gradient descent method. Meanwhile, we get the evolution equations for the level set function:

$$\frac{\mathrm{d}c_i^+(t)}{\mathrm{d}t} = -c_i^+(t)\int_\Omega \beta(x)H(u(t,x))\,\mathrm{d}x + \int_\Omega \beta(x)I_i(x)H(u(t,x))\,\mathrm{d}x, \quad \text{in } \Omega^T, \tag{1a}$$

$$\frac{\mathrm{d}c_i^-(t)}{\mathrm{d}t} = -c_i^-(t)\int_\Omega \beta(x)\big(1 - H(u(t,x))\big)\,\mathrm{d}x + \int_\Omega \beta(x)I_i(x)\big(1 - H(u(t,x))\big)\,\mathrm{d}x, \quad \text{in } \Omega^T, \tag{1b}$$

$$c_i^+(0) = (c_i^+)_0, \qquad c_i^-(0) = (c_i^-)_0, \tag{1c}$$

$$\frac{\partial u}{\partial t} = |\nabla u|\left[\mu \cdot \operatorname{div}\left(g(x)\frac{\nabla u}{|\nabla u|}\right) + \nu \cdot g(x) + \beta(x)\sum_{i=1}^3 \lambda_i^+\big(I_i(x) - c_i^+(t)\big)^2 - \beta(x)\sum_{i=1}^3 \lambda_i^-\big(I_i(x) - c_i^-(t)\big)^2\right], \quad \text{in } \Omega^T, \tag{1d}$$

$$u(0, x) = u_0(x), \quad \text{on } \Omega, \tag{1e}$$

$$\frac{\partial u}{\partial \vec{n}} = 0, \quad \text{on } (\partial\Omega)^T, \tag{1f}$$

where $\vec{n}$ represents the exterior normal to the boundary $\partial\Omega$, $\frac{\partial u}{\partial \vec{n}}$ denotes the normal derivative of u at the boundary, $\Omega^T = [0, T] \times \Omega$, u_0 defines the initial level set, $(c_i^+)_0$ and $(c_i^-)_0$ are the means of intensity for each channel in the area of $u \geqslant 0$ and $u < 0$ respectively, and $H(z)$ denotes the 1D Heaviside function. The detail of the derivation of these equations can be found in [1,8] and [16].

3. The existence of the solution

In this section, we will prove the existence of solution to (1a)–(1f) by using the viscosity theory, the Schauder fixed point theorem and the implicit function theorem in Banach space.

The system of evolution equations is called significative if the following hypotheses are satisfied:

- $\vec{I} = (I_1, I_2, I_3)$, $I_i \geqslant 0$ $(i = 1, 2, 3)$ are Lipschitz functions, i.e., $I_i \in C(\mathbf{R}^2) \cap W^{1,\infty}(\mathbf{R}^2)$;
- The initial level set function $u_0 \in C^\infty(\mathbf{R}^2)$;
- $0 \leqslant (c_i^\pm)_0 \leqslant \max_x I_i(x)$ $(i = 1, 2, 3)$.

Let

$$h(t,x) = \nu g(x) + \beta(x) \sum_{i=1}^{3} \left(\lambda_i^+ \left(I_i(x) - c_i^+(t) \right)^2 - \lambda_i^- \left(I_i(x) - c_i^-(t) \right)^2 \right).$$

In the following, we take $\mu = 1$. The level set evolution equation (1d) is rewritten as

$$\frac{\partial u}{\partial t} = g(x) \sum_{i,j=1}^{2} a_{ij}(\nabla u) u_{x_i x_j} + \sum_{i=1}^{2} \frac{\partial g(x)}{\partial x_i} u_{x_i} + h(t,x)|\nabla u|, \quad \text{in } \Omega^T,$$

where $a_{ij}(p) = \delta_{ij} - \frac{p_i p_j}{|p|^2}$ when $p \neq 0$, and $p \in \mathbf{R}^2$.

Lemma 1. *Let $A = (a_{ij}(p))_{2\times 2}$, $p = (p_1, p_2)$, $a_{ij}(p) = \delta_{ij} - \frac{p_i p_j}{|p|^2}$, then A is a symmetric semi-positive definite matrix.*

Having fixed the function u, we can find the solution $c_i^\pm(t)$ of the ordinary differential equation (ODE) system (1a)–(1c). We have:

Lemma 2. *For all $t \in [0, T]$, $\forall T < \infty$, $c_i^+(t)$, $c_i^-(t)$ $(i = 1, 2, 3)$ are bounded functions, i.e., $0 \leqslant c_i^\pm(t) \leqslant \max_x I_i(x)$.*

Proof. Eqs. (1a)–(1b) are ODEs. So, they can be solved and the solution $c_i^\pm(t) \geqslant 0$. We only prove the property of $c_i^+(t)$. The proof of the $c_i^-(t)$ is the same.

If the maximum of $c_i^+(t)$ is reached at $t = 0$, then $c_i^+(t) \leqslant c_i^+(0)$. By using the hypotheses (c), we have $0 \leqslant c_i^+(t) \leqslant \max_x I_i(x)$.

If the maximum of $c_i^+(t)$ is reached at some $t_0 \in (0, T)$, then $\frac{dc_i^+}{dt}(t_0) = 0$. By (1a), we have

$$c_i^+(t_0) = \frac{\int_\Omega \beta(x) I_i(x) H(u(t_0, x))\, dx}{\int_\Omega \beta(x) H(u(t_0, x))\, dx}.$$

So, $c_i^+(t_0)$ is the mean of intensity on the corresponding area. Therefore, $0 \leqslant c_i^+(t_0) \leqslant \max_x I_i(x)$, and $0 \leqslant c_i^+(t) \leqslant \max_x I_i(x)$.

If the maximum of $c_i^+(t)$ is reached at $t = T$, then there exists a $\delta > 0$ so that Eq. (1a) is still true for all $t \in [0, T + \delta]$ since T is arbitrary, and then the maximum of $c_i^+(t)$ is reached at some interior point. By the above discussion, $0 \leqslant c_i^+(t) \leqslant \max_x I_i(x)$.

If $c_i^+(t)$ strictly increases with t, then $\frac{dc_i^+}{dt}(t) \geqslant 0$. Thus,

$$c_i^+(t) \leqslant \frac{\int_\Omega \beta(x) I_i(x) H(u(t, x))\, dx}{\int_\Omega \beta(x) H(u(t, x))\, dx}.$$

The right-hand side of above equation is the mean of intensity on the corresponding area. Therefore, $0 \leqslant c_i^+(t) \leqslant \max_x I_i(x)$.

In summary, for all $t \in [0, T]$, $\forall T < \infty$, we have $0 \leqslant c_i^+(t) \leqslant \max_x I_i(x)$. $\square$

On the other hand, for fixed $c_i^\pm(t)$, (1d)–(1f) is a partial differential equation system for u. Then, we can prove the existence, uniqueness and stability of viscosity solution to the system (1a)–(1f) by the following theorems.

Theorem 3. *Let $g(x)$ be a smooth function, $I_i \in C(\mathbf{R}^2) \cap W^{1,\infty}(\mathbf{R}^2)$, $I_i(x)$ and $u_0(x)$ are Lipschitz continuous. The problem* (1d)–(1f) *has a viscosity solution $u \in C([0, T] \times \mathbf{R}^2) \cap L^\infty(0, T; W^{1,\infty}(\mathbf{R}^2))$ for any $T < \infty$, and*

$$\inf_{\mathbf{R}^2} u_0(x) \leqslant u(t,x) \leqslant \sup_{\mathbf{R}^2} u_0(x).$$

Theorem 4. *Let $g(x)$ be a smooth function, $I_i \in C(\mathbf{R}^2) \cap W^{1,\infty}(\mathbf{R}^2)$ is Lipschitz continuous, and assume that u_0 is Lipschitz continuous, then the viscosity solution u derived from Theorem 3 is unique. Moreover, if v is a viscosity solution of (1d) with an initial value v_0, then*

$$\sup_{0\leqslant t\leqslant T} \big\| u(t,\cdot) - v(t,\cdot)\big\|_{L^\infty(\mathbf{R}^2)} \leqslant \|u_0 - v_0\|_{L^\infty(\mathbf{R}^2)}$$

for all $T \in [0,\infty)$.

We omit the proof of Theorems 3 and 4 (see [17] and [18]). So from $c_i^{\pm}(t)$ there exists a solution u via (1d)–(1f). From this u we can find a new solution $c_i^{\pm}(t)$ via (1a)–(1c). Then, using this alternative iterative method, a solution $(u, c_i^{\pm})$ of coupling Eqs. (1a)–(1f) could be obtained, provided we can prove the existence of fixed point for the mapping from old $c_i^{\pm}(t)$ to new $c_i^{\pm}(t)$.

For any fixed $T < \infty$, let X be a Banach space defined as follows. It is a product space composed by six continuous function spaces defined on $[0,T]$, $\forall T < \infty$.

$$X = \big\{ f(t) = \big(f_1(t), \dots, f_6(t)\big) \in X \bigm| t \in [0,T],\ \ f_i(0) = 0,\ \ i = 1,\dots,6\big\}$$

with norm

$$\big\| f(x)\big\| = \max_{t\in[0,T],\ i=1,\dots,6} \big| f_i(t)\big|.$$

Let $d_i^{\pm}(t) = c_i^{\pm}(t) - (c_i^{\pm})_0$, then $d_i^{\pm}(0) = 0$. We have $-(c_i^{\pm})_0 \leqslant d_i^{\pm}(t) \leqslant \max_x I_i(x) - (c_i^{\pm})_0$ since $0 \leqslant c_i^{\pm}(t) \leqslant \max_x I_i(x)$. We choose a proper closed convex subset Y in X,

$$Y = \big\{ d(t) = \big(d_i^+(t), d_i^-(t)\big) \in X \bigm| t \in [0,T],\ \ -\big(c_i^{\pm}\big)_0 \leqslant d_i^{\pm}(t) \leqslant \max_x I_i(x) - \big(c_i^{\pm}\big)_0,\ \ i = 1,2,3\big\}.$$

Accordingly, its norm is

$$\big\| d(t)\big\| = \max\Big\{ \max_{t,i}\big|d_i^+(t)\big|, \max_{t,i}\big|d_i^-(t)\big|\Big\}.$$

Let $c_0 = ((c_i^+)_0, (c_i^-)_0)$. For a set of $c(t) = d(t) + c_0$, the system (1d)–(1f) has a solution $u(t,x)$ by Theorem 3. Solving (1a)–(1c), we get a new set of $\tilde{c}(t) = \tilde{d}(t) + c_0$. From Lemma 2, $d(t), \tilde{d}(t) \in Y$. We define a mapping

$$\mathcal{T}: Y \longrightarrow Y,$$
$$d(t) \longmapsto \tilde{d}(t).$$

In order to prove the existence of fixed point for the map $\mathcal{T}$ using Schauder fixed point theorem, we must verify that the assumptions for the Schauder fixed point theorem. The following two lemmas guarantee the assumptions.

Lemma 5. *If $\mathcal{T}: Y \to Y$, then $\mathcal{T}Y \subset Y$, and $\mathcal{T}Y$ is precompact.*

Proof. By the definition of the mapping $\mathcal{T}$ and Lemma 2, it is clear that $\mathcal{T}Y \subset Y$.

For any sequence $\{d_m(t) = ((d_i^+)_m(t), (d_i^-)_m(t))\}$ in $\mathcal{T}Y$, $(d_i^{\pm})_m(t)$ is bounded by Lemma 2 and the facts that $0 \leqslant \beta(x) \leqslant 1$, $0 \leqslant H(z) \leqslant 1$ and $0 \leqslant I_i(x) \leqslant 255$. Therefore, $|\frac{\mathrm{d}(d_i^{\pm})_m(t)}{\mathrm{d}t}|$ is bounded. Moreover, there exists a positive constant M, which depends only on $\max_x I_i(x)$ and the volume of Ω, such that $|d_m'(t)| \leqslant M$. For any $t, s \in [0,T]$, we have

$$\big| d_m(t) - d_m(s)\big| \leqslant M|t-s|.$$

There is a subsequence $\{d_{m_k}(t)\}$ of $\{d_m(t)\}$ by the Arzelà–Ascoli theorem, such that $\{d_{m_k}(t)\}$ converges to some $d(t)$ in Y as $k \to \infty$. Thus, $\mathcal{T}Y$ is precompact. □

Lemma 6. *$\mathcal{T}$ is a continuous mapping.*

Proof. We hope that u will have a small increment if $c(t) = d(t) + c_0$ has a small increment, since u is an intermediate variable by the definition of $\mathcal{T}$. The solution of ODEs (1a) and (1b), $\tilde{c}(t) = \tilde{d}(t) + c_0$ will have a corresponding small increment, since the solution of ODE continuously depends on the coefficients. So, we are able to prove the continuity of the mapping $\mathcal{T}$. Therefore, the key to this problem is to prove that $u(t,x)$ continuously depends on $d(t)$ in the regularized equations of (1d)–(1f).

Let $u^\epsilon(t,x) = v(t,x) + u_0^\epsilon(x)$, $b^\epsilon(\nabla(v+u_0^\epsilon)) = \sqrt{|\nabla(v+u_0^\epsilon)|^2+\epsilon}$. Eqs. (1d)–(1f) are rewritten as the equations about $v(t,x)$:

$$\begin{cases} \dfrac{\partial(v+u_0^\epsilon)}{\partial t} - g(x)\displaystyle\sum_{i,j=1}^{2} a_{ij}^\epsilon\big(\nabla v + \nabla u_0^\epsilon\big)\big(v+u_0^\epsilon\big)_{x_i x_j} - \sum_{i=1}^{2}\frac{\partial g(x)}{\partial x_i}\big(v+u_0^\epsilon\big)_{x_i} - b^\epsilon\big(\nabla\big(v+u_0^\epsilon\big)\big) \\ \quad\times\left[\nu g(x) + \beta(x)\displaystyle\sum_{i=1}^{3}\lambda_i^+\big(I_i(x) - \big(d_i^+(t) + \big(c_i^+\big)_0\big)\big)^2 - \beta(x)\sum_{i=1}^{3}\lambda_i^-\big(I_i(x) - \big(d_i^-(t) + (c_i^-)_0\big)\big)^2\right] = 0 \quad \text{in } \Omega^T, \\ v(0,x) = 0 \quad \text{on } \Omega, \\ \dfrac{\partial(v+u_0^\epsilon)}{\partial\vec{n}} = 0 \quad \text{on } (\partial\Omega)^T. \end{cases}$$

We will use the implicit function theorem in Banach space to prove that $v(t,x)$ continuously depends on $d(t)$.

Let

$$D(t) = \big(D_1^+(t), D_2^+(t), D_3^+(t), D_1^-(t), D_2^-(t), D_3^-(t)\big),$$

where $t \in [0,T]$ and $D(0)=0$. Let E_1 be a space composed by all these functions. E_2 is composed by all continuous functions $V(t,x)$ with initial value 0. F is a continuous functional space defined on Ω^T. So, E_1, E_2 and F are Banach spaces. Define

$$f : E_1 \times E_2 \longrightarrow F,$$
$$\big(D(t), V(t,x)\big) \longmapsto f\big(D(t), V(t,x)\big)$$

where

$$f\big(D(t),V(t,x)\big) = \frac{\partial(V+u_0^\epsilon)}{\partial t} - g(x)\sum_{i,j=1}^{2} a_{ij}^\epsilon\big(\nabla V + \nabla u_0^\epsilon\big)\big(V+u_0^\epsilon\big)_{x_i x_j} - \sum_{i=1}^{2}\frac{\partial g(x)}{\partial x_i}\big(V+u_0^\epsilon\big)_{x_i} - b^\epsilon\big(\nabla\big(V+u_0^\epsilon\big)\big)$$
$$\times\left[\nu g(x) + \beta(x)\sum_{i=1}^{3}\lambda_i^+\big(I_i(x) - \big(D_i^+(t) + \big(c_i^+\big)_0\big)\big)^2 - \beta(x)\sum_{i=1}^{3}\lambda_i^-\big(I_i(x) - \big(D_i^-(t) + (c_i^-)_0\big)\big)^2\right].$$

It is clear that $f(d(t), v(t,x)) = 0$ when $(D(t), V(t,x)) = (d(t), v(t,x))$.

The partial differentiation $(D_2 f) : E_2 \to F$ at $(d(t), v(t,x))$ is just a linearization of $f(D(t), V(t,x))$ about $V(t,x)$ at the point $(d(t), v(t,x))$. Thus, we calculate $\frac{\mathrm{d}}{\mathrm{d}s} f(d(t), v(t,x) + sw(t,x))\big|_{s=0}$. By complicated computation, we have

$$\frac{\mathrm{d}}{\mathrm{d}s} f\big(d(t), v(t,x) + sw(t,x)\big)\bigg|_{s=0} = \frac{\partial w}{\partial t} - \sum_{i,j=1}^{2} A_{ij}(t,x) w_{x_i x_j} + \sum_{i=1}^{2} B_i(t,x) w_{x_i},$$

where $A_{ij}(t,x)$, $B_i(t,x)$ are represented by the known functions $v(t,x)$, $c_i^\pm(t)$ and $g(x)$. $A_{ij}(t,x)$ is a positive matrix, and all its eigenvalues are not smaller than some constant $\epsilon > 0$. Therefore, $(D_2 f) : E_2 \to F$ is a linear mapping

$$\mathcal{L} : E_2 \longrightarrow F,$$
$$w \longmapsto \mathcal{L}w,$$

where

$$\mathcal{L}w = \frac{\partial w}{\partial t} - \sum_{i,j=1}^{2} A_{ij}(t,x) w_{x_i x_j} + \sum_{i=1}^{2} B_i(t,x) w_{x_i}.$$

In order to prove the continuity of the mapping $\mathcal{T}$, we must verify that $\mathcal{L}$ is an isomorphic mapping, i.e., for any $f \in F$, there exists a $w \in E_2$ such that $\mathcal{L}w = f$.

Since $w \in E_2$ satisfies $w(0,x) = 0$, w is the solution of the second order parabolic equations

$$\begin{cases} \mathcal{L}w = f & \text{on } \Omega^T, \\ w|_{t=0} = 0 & \text{on } \Omega, \\ \dfrac{\partial w}{\partial \vec{n}} = 0 & \text{on } (\partial\Omega)^T. \end{cases}$$

By the classical theory of second order parabolic equations, there exists a solution $w(t,x)$ that satisfies $w(0,x) = 0$ with Neumann boundary condition, and $\max_{\Omega^T} |w(t,x)| \leqslant C \cdot \max_{\Omega^T} |f(t,x)|$, where the constant C depends only on the known coefficients of the parabolic operator $\mathcal{L}$. Therefore, $w(t,x)=0$ as $f(t,x)=0$. So, the mapping $\mathcal{L}: E_2 \to F$ is an isomorphic. As a result, $\mathcal{T}: Y \to Y$ is a continuous mapping. □

Then, we obtain our main theorem:

Main Theorem. *The solution of the system of Eqs.* (1a)–(1f) *exists.*

Proof. By Lemmas 5 and 6, there exists a fixed point $d(t) \in Y$, such that $\mathcal{T}(d(t)) = d(t)$, i.e., $\tilde{c}(t) = c(t)$. Therefore, the solution of the system of evolution Eqs. (1a)–(1f) exists by Theorems 3 and 4. □

4. Implementation details and experimental results

We use a finite difference scheme to discretized Eqs. (1a)–(1f) [14,15]. For simplicity, we write (x, y) for the spatial variables $(x_1, x_2) \in \Omega \subset \mathbf{R}^2$. We denote the space step by $h=1$ and the time step by τ. Then $(x_i, y_j) = (ih, jh)$, $1 \leqslant i, j \leqslant M$, and $t_n = n\tau$, where M is the image size. Thus, we denote $u_{ij} = u(t_n, ih, jh)$. We mainly present the scheme of the equation of level set (1d):

$$\frac{\partial u}{\partial t} = \mu \cdot g(x)|\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) + \mu \nabla g(x) \cdot \nabla u + |\nabla u| \left[\nu g(x) + \beta(x) \sum_{i=1}^{3} \lambda_i^+ \left(I_i(x) - c_i^+\right)^2 - \beta(x) \sum_{i=1}^{3} \lambda_i^- \left(I_i(x) - c_i^-\right)^2 \right],$$

$$\left(\nabla g(x) \cdot \nabla u\right)_{ij}^n = \max\left(D^x g_{ij}, 0\right) D^{-x} u_{ij}^n + \min\left(D^x g_{ij}, 0\right) D^{+x} u_{ij}^n + \max\left(D^y g_{ij}, 0\right) D^{-y} u_{ij}^n + \min\left(D^y g_{ij}, 0\right) D^{+y} u_{ij}^n,$$

where

$$D^{-x}u_{ij}^n = u_{i,j} - u_{i-1,j}, \qquad D^{+x}u_{ij}^n = u_{i+1,j} - u_{i,j}, \qquad D^x u_{ij}^n = \frac{u_{i+1,j} - u_{i-1,j}}{2},$$
$$D^{-y}u_{ij}^n = u_{i,j} - u_{i,j-1}, \qquad D^{+y}u_{ij}^n = u_{i,j+1} - u_{i,j}, \qquad D^y u_{ij}^n = \frac{u_{i,j+1} - u_{i,j-1}}{2}.$$

The upwind scheme is used to $|\nabla u|$ [14]:

$$|\nabla u|_{ij}^n \approx \begin{cases} |\nabla^- u|_{ij}^n = (\max(D^{-x}u_{ij}^n, 0)^2 + \min(D^{+x}u_{ij}^n, 0)^2 + \max(D^{-y}u_{ij}^n, 0)^2 + \min(D^{+y}u_{ij}^n, 0)^2)^{\frac{1}{2}}, & \nu \leqslant 0, \\ |\nabla^+ u|_{ij}^n = (\min(D^{-x}u_{ij}^n, 0)^2 + \max(D^{+x}u_{ij}^n, 0)^2 + \min(D^{-y}u_{ij}^n, 0)^2 + \max(D^{+y}u_{ij}^n, 0)^2)^{\frac{1}{2}}, & \nu > 0. \end{cases}$$

We denote

$$F(x, y) \triangleq \nu g(x, y) + \frac{\beta(x, y)}{3} \sum_{i=1}^{3} \lambda_i^+ \left(I_i(x, y) - c_i^+\right)^2 - \frac{\beta(x, y)}{3} \sum_{i=1}^{3} \lambda_i^- \left(I_i(x, y) - c_i^-\right)^2.$$

The finite difference scheme of the level set equation is

$$u_{ij}^{n+1} = u_{ij}^n + \tau\left[\mu g_{ij} k_{ij}^n \left(\left(D^x u_{ij}^n\right)^2 + \left(D^y u_{ij}^n\right)^2\right)^{\frac{1}{2}} + \mu\left(\nabla g(x) \cdot \nabla u\right)_{ij}^n + \max\left(F_{ij}^n, 0\right)|\nabla^+ u|_{ij}^n + \min\left(F_{ij}^n, 0\right)|\nabla^- u|_{ij}^n\right],$$

where k_{ij}^n is the central finite difference approximation of the curvature:

$$k = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = \frac{u_{xx}u_y^2 - 2u_x u_y u_{xy} + u_{yy}u_x^2}{(u_x^2 + u_y^2)^{\frac{3}{2}}}.$$

The experimental results are shown in Figs. 1 and 2. Our model only detects the desired object(s), while the traditional Chan–Vese methods will detect all the objects in a given image. In all the experiments in this paper, the τ is set as 5, $\sigma = 0.5$. and $\lambda^{\pm} = (1, 1, 1)$.

Fig. 1 shows an example where our model detects only the clouds. We choose six sample pixels in the cloud area, and the parameters are $\mu = 1$ and $\nu = -100$.

Fig. 2 gives an example to show that we can segment different desired object(s) by choosing sample pixels from different areas. In the first result the desired object was the rubber ring. In this experiment, five sample pixels were chosen in the rubber ring, and the parameters used were $\mu = 0.1$ and $\nu = -10$. In the second result the desired object was the hand. We chose seven pixels in the hand area. The parameters used were $\mu = 1$ and $\nu = 0$.

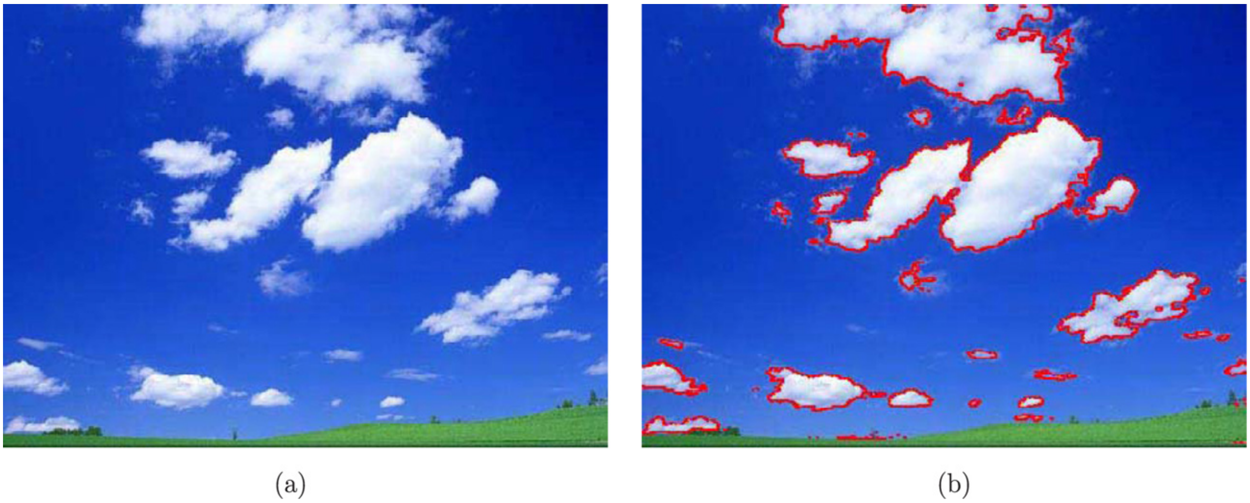

Fig. 1. Result for the detection of clouds. (a) The initial image; (b) The result.

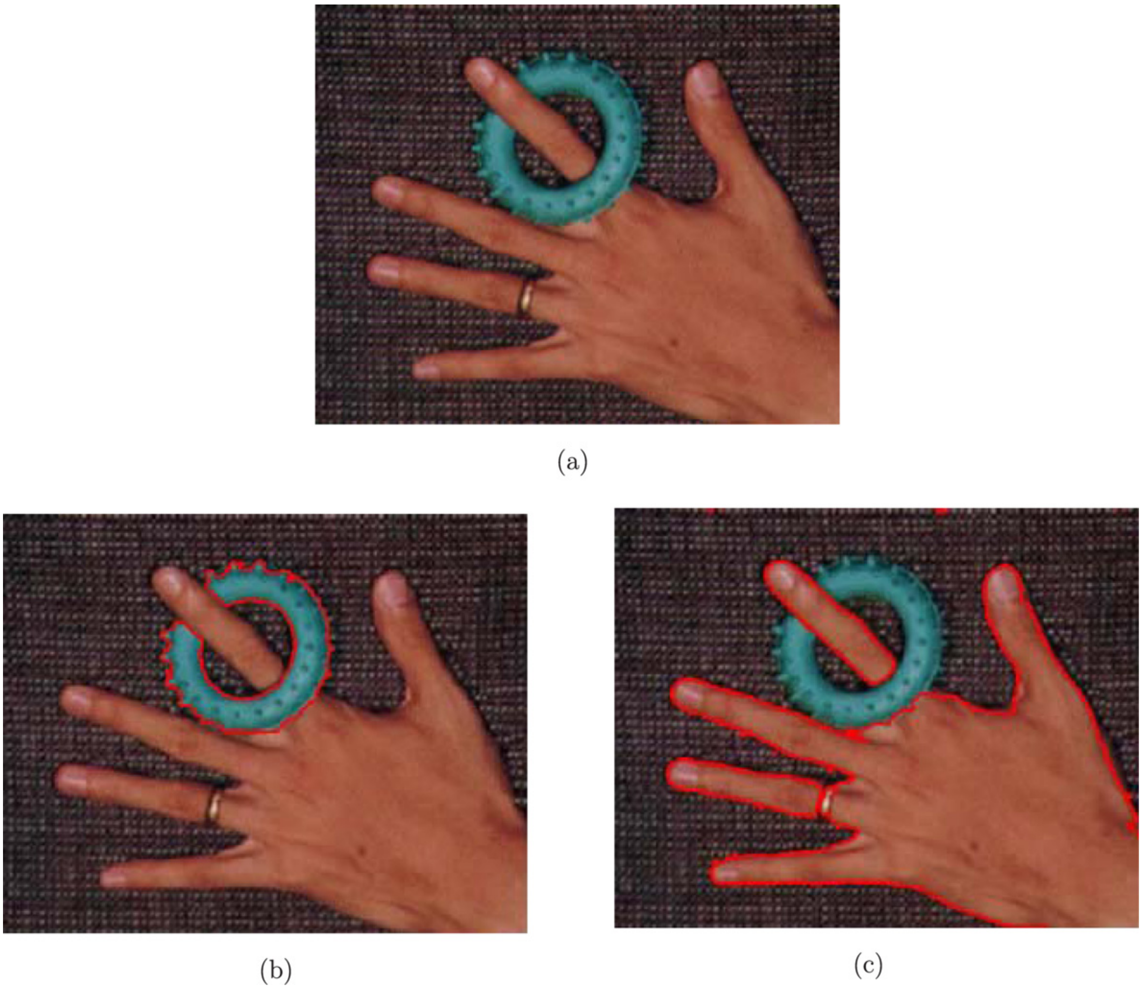

Fig. 2. (a) The initial image; (b) The result of detecting the rubber ring; (c) The result of detecting the hand.

5. Conclusion

We have proposed a modified Chan–Vese model to detect the interior and exterior boundaries of the desired object(s) in color images. Two real color images are used to examine our method, and the results are found satisfactory as only the desired object(s) are accurately delineated. Moreover, the existence of the solution to the evolution equations has been proved.

References

[1] T. Chan, L. Vese, Active contour without edges, IEEE Trans. Image Process. 10 (2001) 266–277.
[2] T. Chan, B.Y. Sandberg, L. Vese, Active contour without edges for vector-valued image, J. Vis. Commun. Image Represent. 11 (2001) 130–141.
[3] L. Pi, J.S. Fan, C.M. Shen, Color image segmentation for objects of interest with modified geodesic active contour method, J. Math. Imaging Vision 27 (2007) 51–57.
[4] S. Osher, R. Fedkiw, Level Set Methods and Dynamic Implicit Surfaces, Springer-Verlag, New York, 2002.
[5] M.G. Crandall, H. Ishii, P.L. Lions, User's guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc. 27 (1992) 1–67.
[6] A. Friedman, Partial Differential Equations of Parabolic Type, Prentice Hall, Inc., 1964.
[7] S.K. Donaldson, P.B. Kronheimer, The Geometry of Four-Manifolds, Clarendon Press, Oxford, 1990.
[8] V. Caselles, R. Kimmel, G. Sapiro, Geodesic active contours, Int. J. Comput. Vision 22 (1997) 61–79.
[9] S. Kichenesamy, A. Kumar, P. Olver, A. Tannenbaum, A. Yezzi, Gradient flows and geometric active contours, in: Proc. of International Conference on Computer Vision, 1995.
[10] D. Mumford, J. Shah, Optimal approximations by piecewise smooth functions and associated variational problems, Comm. Pure Appl. Math. 42 (1989) 577–685.
[11] R. Kimmel, Fast edge integration, in: S. Osher, N. Paragios (Eds.), Geometric Level Set Methods in Imaging, Vision and Graphics, Springer-Verlag, New York, 2003.
[12] R. Goldenberg, R. Kimmel, E. Rivlin, M. Rudzsky, Fast geodesic active contours, IEEE Trans. Image Process. 10 (2001) 1467–1475.
[13] T.W. Anderson, An Introduction to Multivariate Statistical Analysis, second ed., Wiley, New York, 1984.
[14] S.J. Osher, J.A. Sethian, Fronts propagating with curvature dependent speed: Algorithms based on Hamilton–Jacobi formulations, J. Comput. Phys. 79 (1988) 12–49.
[15] J.A. Sethian, A review of recent numerical algorithms for hypersurfaces moving with curvature dependent flows, J. Differential Geom. 31 (1989) 131–161.
[16] V. Caselles, F. Catte, T. Coll, F. Dibos, A geometric model for active contours in image processing, Numer. Math. 66 (1993) 1–31.
[17] L. Alvarez, P.L. Lions, J.M. Morel, Image selective smoothing and edge detection by nonlinear diffusion (II), SIAM J. Numer. Anal. 29 (1992) 845–866.
[18] G. Aubert, P. Kornprobst, Mathematical Problems in Image Processing: Partial Differential Equations and the Calculus of Variations, Springer-Verlag, New York, 2002.