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Variable exponent functionals in image restoration

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ABSTRACT

We study a functional with variable exponent, $1 < p(x) \leqslant 2$, which provides a model for image denoising and restoration. Here $p(x)$ is defined by the gradient information in the observed image. The diffusion derived from the proposed model is between total variation based regularization and Gaussian smoothing. The diffusion speed of the corresponding heat equation is tuned by the variable exponent $p(x)$. The minimization problem and its associated flow in a weakened formulation are discussed. The existence, uniqueness, stability and long-time behavior of the proposed model are established in the variable exponent functional space $W^{1,p(x)}$. Experimental results illustrate the effectiveness of the model in image restoration.

1. Introduction

Image denoising is one of the fundamental problems in image processing with numerous applications. The aim of image denoising is to design methods which can selectively smooth a noisy image without losing significant features such as edges.

Variational denoising methods are widely studied numerically and theoretically in recent years. In variational framework, the denoising problem can be expressed as follows: given an original image *f*, it is assumed that it has been corrupted by some additive noise *n*. Then the problem is to recover the true image *u* from

$$f = u + n.$$

Let us consider the following representative minimization problem

$$\min\left\{E(u) = \int_\Omega |\nabla u|^p dx + \frac{\lambda}{2}\int_\Omega (u-f)^2 dx\right\}, \tag{1.1}$$

where $1 \leqslant p \leqslant 2$ is a constant and λ is a scalar parameter. The first term in the energy functional of (1.1) is a regularization term and the second term is a fidelity term. As $p = 1$, it is the widely used Rudin–Osher–Fatemi (ROF) model proposed in 1992 [12]. The considerable advantage of the ROF model is that it can well preserve edge sharpness and location while smooth out noise. Mathematically, it is reasonable since its solution belongs to bounded variation (BV) space which allows discontinuities in functions. However, the ROF model favors solutions that are piecewise constant which often causes the *staircasing* effect [11,14,15]. The staircasing effect creates *false* edges which are misleading and not satisfactory in visual effects.

Choosing $p = 2$ in (1.1) results in isotropic diffusion which solves the staircasing effect problem but it oversmoothes images such that the edges are blurred and dislocated. A fixed value of $1 < p < 2$ results in anisotropic diffusion between

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the ROF model and the isotropic smoothing. However, there is a trade-off between piecewise smooth regions reconstruction and edge preservation.

Since different values of p should have different advantages, it encourages one to combine their benefits with a variable exponent. Blomgren et al. proposed the following minimization problem in [1]

$$\min\{E(u) = \int_\Omega |\nabla u|^{p(|\nabla u|)} dx\}, \tag{1.2}$$

where $\lim_{s\to 0} p(s) = 2, \lim_{s\to\infty} p(s) = 1$, and p is a monotonically decreasing function. This model is a variable exponent model. It chooses diffusion speed through exponent and then can reduce the staircasing effect. Since p depends on ∇u, it is hard to establish the lower semi-continuity of the energy functional. Bollt et al. proved that this problem with an L^1 or L^2 norm fidelity term has a minimizer in [2], however, nothing about the associated heat equations was discussed.

Later, Chen et al. proposed the following model in [3]

$$\min_{u\in BV(\Omega)\cap L^2(\Omega)} \left\{ E(u) = \int_\Omega \varphi(x, Du) + \frac{\lambda}{2}\int_\Omega (u-f)^2 dx \right\}, \tag{1.3}$$

where

$$\varphi(x,r) = \begin{cases} \frac{1}{q(x)}|r|^{q(x)}, & |r| \leqslant \beta, \\ |r| - \frac{\beta q(x) - \beta^{q(x)}}{q(x)}, & |r| > \beta, \end{cases}$$

$q(x) = 1 + \frac{1}{1+k|\nabla G_\sigma * f(x)|}$, $G_\sigma(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(\frac{|x|^2}{2\sigma^2}\right)$ is the Gaussian kernel, $k > 0$, $\sigma > 0$ are fixed parameter, and β is a user-defined threshold. Mathematically, the energy minimization problem and the associated heat flow were discussed.

Inspired by the above models, we propose the model

$$\min_{u\in W^{1,p(x)}(\Omega)\cap L^2(\Omega)} \left\{ E_{p(x)}(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)} dx + \frac{\lambda}{2}\int_\Omega (u-f)^2 dx \right\}, \tag{1.4}$$

where $p(x) = 1 + g(x)$ and $g(x) = \frac{1}{1+k|\nabla G_\sigma * f(x)|}$.

Clearly in the regions with edges, $g \to 0$ since the image gradient is large, model (1.4) approximates the ROF model, so the edges will be preserved; In relatively smooth regions $g \to 1$ since image gradient is small, model (1.4) approximates isotropic smoothing, so they will be processed into piecewise smooth regions. In other regions, the diffusion is properly adjusted by the function $p(x)$.

The proposed model (1.4) is simpler than (1.3) in the formulation. Meanwhile, model (1.4) is more automatic than (1.3) since no user-defined threshold β is needed in (1.4). Chen et al. studied problem (1.3) in BV framework [3], however, in this paper we will study problem (1.4) in the variable exponent space $W^{1,p(x)}$.

The paper is organized as follows: in Section 2 we give some important lemmas and then prove the existence and uniqueness of the solution of the minimization problem (1.4). In Section 3 we prove the existence, uniqueness and stability of the solution of the heat flow problem and discuss the long-time behavior. In Section 4 we provide our numerical algorithm and experimental results to illustrate the effectiveness of our model in image restoration. Finally, we conclude the paper in Section 5.

2. The minimization problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, $f \in L^\infty(\Omega)$. By the definition of $g(x)$ and Gaussian convolution, we obtain $\nabla G_\sigma * f \in C^\infty(\Omega)$. Then there exists a constant $M > 0$, such that $|\nabla G_\sigma * f| \leqslant M$. Therefore, $g(x) \geqslant \frac{1}{1+M^2}$ and $p(x) \geqslant 1 + \frac{1}{1+M^2} > 1$. Meanwhile, since $g(x) \leqslant 1$, we get $1 < p(x) \leqslant 2$ in the proposed model (1.4).

Variable exponent spaces. Let $p(x) : \Omega \to [1, +\infty)$ be a measurable function, called variable exponent on Ω. By $\mathscr{P}(\Omega)$ we denote the family of all measurable functions on Ω. Let $p^- := \operatorname{ess\,inf}_\Omega p(x), p^+ := \operatorname{ess\,sup}_\Omega p(x)$. We define a functional

$$Q_{p(x)}(u) = \int_\Omega |u|^{p(x)} dx$$

and a norm by formula

$$\|u\|_{p(x)} = \|u\|_{L^{p(x)}(\Omega)} := \inf\{\lambda > 0 : Q_{p(x)}(u/\lambda) \leqslant 1\}.$$

Then the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ and the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ are defined as

$$L^{p(x)}(\Omega) = \{u : \Omega \to \mathbb{R} | \|u\|_{p(x)} < \infty\},$$
$$W^{1,p(x)}(\Omega) = \{u : \Omega \to \mathbb{R} | u \in L^{p(x)}(\Omega), \nabla u \in L^{p(x)}(\Omega)\}.$$

With the norm $\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}$, $W^{1,p(x)}(\Omega)$ becomes a Banach space. $W_0^{1,p(x)}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ under the norm $\|\cdot\|_{1,p(x)}$. See [4] for the basic theory of variable exponent spaces.

In the following, we cite Lemmas 2.1 and 2.2 from [8]. Then we prove Lemmas 2.3–2.5 as the preparation for the proof of the main theorems.

Lemma 2.1. *Let $p(x), q(x) \in \mathscr{P}(\Omega)$, and for a.e. $x \in \Omega$ we have $p(x) \leqslant q(x)$. Then $L^{q(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega), W^{1,q(x)}(\Omega) \hookrightarrow W^{1,p(x)}(\Omega)$. The norm of the embedding operator does not exceed $1 + |\Omega|$, where $|\Omega|$ denotes the measure of Ω.*

Lemma 2.2. *Let $p(x) \in \mathscr{P}(\Omega)$, $1 < p^- \leqslant p^+ < \infty$. Then $L^{p(x)}(\Omega), W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are all reflexive Banach spaces.*

Lemma 2.3. *Let $F(\nabla u, u, x) = \frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{\lambda}{2}(u-f)^2, p(x) = 1 + g(x)$ as in model (1.4). Then for each z, x, $F(\xi, z, x)$ is convex in ξ.*

Proof. As we know, if a multivariable function $G(x), x = (x_1, \ldots, x_n)$ is twice differentiable, then G is convex if and only if the Hessian matrix $\nabla^2 G(x) = \frac{\partial^2 G}{\partial x_i \partial x_j}(x)$ is semi-positively definite for abitrary $x \in \mathrm{dom}(G)$. Let $F(\xi, z, x) = \frac{1}{p(x)}|\xi|^{p(x)} + \frac{\lambda}{2}(z-f)^2$. Then

$$F_{\xi_i} = \frac{1}{p(x)} p(x)|\xi|^{p(x)-1}\frac{\xi_i}{|\xi|} = |\xi|^{p(x)-2}\xi_i,$$
$$F_{\xi_i\xi_j} = (p(x)-2)|\xi|^{p(x)-4}\xi_i\xi_j + |\xi|^{p(x)-2}\delta_{ij},$$
$$F_{\xi_i\xi_j}\eta^i\eta^j = (p(x)-2)|\xi|^{p(x)-4}\xi_i\xi_j\eta^i\eta^j + |\xi|^{p(x)-2}\delta_{ij}\eta^i\eta^j, \quad \forall \eta \in R^n,$$

where the same upper and lower index denotes summation from 1 to n. By Cauchy inequality

$$\xi_i\xi_j\eta^i\eta^j = \left(\sum \xi_i\eta_i\right)^2 \leqslant \left(\sum \xi_i^2\right)\left(\sum \eta_i^2\right) = |\xi|^2|\eta|^2$$

and the condition $1 < p(x) \leqslant 2$, we obtain

$$F_{\xi_i\xi_j}\eta^i\eta^j \geqslant (p(x)-2)|\xi|^{p(x)-4}|\xi|^2|\eta|^2 + |\xi|^{p(x)-2}|\eta|^2 = (p(x)-1)|\xi|^{p(x)-2}|\eta|^2 \geqslant 0.$$

Therefore, F is convex in ξ. □

Lemma 2.4. *Let $F(\xi, z, x)$ be bounded from below, and the map $\xi \mapsto F(\xi, z, x)$ is convex in each $z \in R, x \in \Omega$. Then the energy functional $I(u) := \int_\Omega F(\nabla u, u, x)dx$ is weakly lower semi-continuous in $W^{1,p(x)}(\Omega)$.*

Mimicking the proof of Theorem 1 (p. 446 in [4]) in which p is a constant, we can prove the variable exponent case which is Lemma 2.4. In the proof, we need the Sobolev embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$. Fortunately, under the assumption of this paper, the Sobolev embedding holds. It is the following lemma.

Lemma 2.5. *Let the dimension of Ω be $n = 2, 1 \leqslant p^- \leqslant p(x) \leqslant p^+ \leqslant 2$. Then the embedding $W^{1,p(x)} \hookrightarrow L^{p(x)}$ is compact.*

Proof. From $n = 2$, we deduce that $(p^-)^* = \frac{np^-}{n-p^-} = \frac{n}{n/p^- - 1} \geqslant \frac{n}{n-1} = 2 \geqslant p^+$. Since Ω is bounded open set with Lipschitz boundary, by the Sobolev embedding theorem (where p is constant) in [4] and Lemma 2.1, we obtain

$$W^{1,p(x)}(\Omega) \hookrightarrow W^{1,p^-}(\Omega) \hookrightarrow L^{p+}(\Omega) \hookrightarrow L^{p(x)}(\Omega),$$

where the embedding $W^{1,p^-}(\Omega) \hookrightarrow L^{p+}(\Omega)$ is compact. Therefore, the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$ is compact.

Note that in our model (1.4), $p(x) = 1 + g(x)$. By definition of $g(x)$, we have $1 < p^- \leqslant p(x) \leqslant p^+ \leqslant 2$, which satisfies the condition of Lemma 2.5. Similar results are also established in [5,6]. □

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with Lipschitz boundary, $f \in W^{1,p(x)}(\Omega) \cap L^2(\Omega)$. Then the minimization problem*

$$\min_{u \in W^{1,p(x)}(\Omega) \cap L^2(\Omega)} \left\{ E_{p(x)}(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \frac{\lambda}{2}\int_\Omega (u-f)^2 dx \right\}$$

has a unique minimizer $u \in W^{1,p(x)}(\Omega) \cap L^2(\Omega)$.

Proof. Let $\mu = \inf_{v \in W^{1,p(x)}(\Omega) \cap L^2(\Omega)} E_{p(x)}(v)$. Since $f \in W^{1,p(x)}(\Omega) \cap L^2(\Omega)$, μ is finite. Let $\{u_k\}_{k=1}^\infty, u_k \in W^{1,p(x)}(\Omega) \cap L^2(\Omega)$ be the minimizing sequence such that $E_{p(x)}(u_k) \to \mu$. Then there exists a constant C, such that

$$\int_\Omega \frac{1}{p(x)}|\nabla u_k|^{p(x)}dx \leqslant C \quad \text{and} \quad \int_\Omega (u_k - f)^2 dx \leqslant C.$$

Hence $\int_\Omega (u_k)^2 dx \leqslant C$. By Lemma 2.1, $L^2(\Omega) \subset L^{p(x)}(\Omega)$. So we have $\int_\Omega |u|^{p(x)}dx \leqslant C$.
Together with the inequality

$$\int_\Omega |\nabla u_k|^{p(x)}dx \leqslant C\int_\Omega \frac{1}{p^+}|\nabla u_k|^{p(x)}dx \leqslant C\int_\Omega \frac{1}{p(x)}|\nabla u_k|^{p(x)}dx \leqslant C,$$

we obtain $Q_{p(x)}(u_k) + Q_{p(x)}(\nabla u_k) \leqslant C$. This implies that $\{u_k\}_{k=1}^{\infty}$ is a uniformly bounded sequence in $W^{1,p(x)}(\Omega)$. Meanwhile, $\{u_k\}_{k=1}^{\infty}$ is uniformly bounded in $L^2(\Omega)$. Since $W^{1,p(x)}(\Omega) \cap L^2(\Omega)$ is a reflexive Banach space, there exists a subsequence $\{u_{k_j}\}_{j=1}^{\infty} \subset \{u_k\}_{k=1}^{\infty}$, and a function $u \in W^{1,p(x)}(\Omega) \cap L^2(\Omega)$, such that

$$u_{k_j} \rightharpoonup u \text{ in } W^{1,p(x)}(\Omega) \cap L^2(\Omega).$$

By Lemma 2.4, $E_{p(x)}$ is weakly lower semi-continuous in $W^{1,p(x)}(\Omega) \cap L^2(\Omega)$. Then we have

$$E_{p(x)}(u) \leqslant \liminf_{j\to\infty} E_{p(x)}(u_{k_j}) = \mu.$$

Therefore, u is a minimizer of $E_{p(x)}$. The uniqueness follows from the strict convexity of $E_{p(x)}(u)$ about u. □

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with Lipschitz boundary, $w \in W^{1,p(x)}(\Omega) \cap L^{\infty}(\Omega), f - w \in W_0^{1,p(x)}(\Omega) \cap L^{\infty}(\Omega)$. Then the minimization problem*

$$\min_{u \in W^{1,p(x)}(\Omega)\cap L^2(\Omega),\, u-w\in W_0^{1,p(x)}} \left\{ E_{p(x)}(u) = \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \frac{\lambda}{2} \int_{\Omega} (u-f)^2 dx \right\} \tag{2.1}$$

has a unique minimizer $u \in W^{1,p(x)}(\Omega) \cap L^{\infty}(\Omega)$, which satisfies $u - w \in W_0^{1,p(x)}(\Omega) \cap L^{\infty}(\Omega)$.

Proof. Let $u \in U(= W^{1,p(x)}(\Omega) \cap L^2(\Omega))$ and denote $a = \max\{\|w\|_{\infty}, \|f\|_{\infty}\}$. Let u_a be the function u which has been cut-off at $-a$ and a, i.e. $u_a = \min\{a, \max\{-a, u\}\}$. By definition of a, it is easy to see that $u_a - w \in W_0^{1,p(x)}(\Omega) \cap L^2(\Omega)$. Moreover,

$$\nabla u_a = \begin{cases} \nabla u, & |u| \leqslant a, \\ 0, & |u| > a. \end{cases}$$

Hence $|\nabla u_a| \leqslant |\nabla u|$ *a.e.* $x \in \Omega$ and so $E_{p(x)}(u_a) \leqslant E_{p(x)}(u)$. It follows that it suffices to look for minimizers in the set $U_a = \{u_a : u \in U\}$.

Let $\mu = \inf_{v \in U_a} E_{p(x)}(v)$, and $\{u_k\}_{k=1}^{\infty} \subset U_a$ be a minimizing sequence. Then $E_{p(x)}(u_k) \to \mu$. Hence

$$\int_{\Omega} \frac{1}{p(x)} |\nabla u_k|^{p(x)} dx \leqslant C, \quad \int_{\Omega} (u_k - f)^2 dx \leqslant C.$$

By the $W^{1,1}$ – Sobolev–Poincaré inequality, the embedding $L^{p(x)}(\Omega) \hookrightarrow L^1(\Omega)$ and the fact $u_k - w \in W_0^{1,1}(\Omega)$, we get

$$\int_{\Omega} |u_k|^{p(x)} dx = \int_{\Omega} |u_k|^{p(x)-1} |u_k| dx \leqslant a^{p^+-1} \int_{\Omega} |u_k| dx \leqslant C \int_{\Omega} |u_k - w| + |w| dx \leqslant C \int_{\Omega} |\nabla u_k - \nabla w| dx + C \leqslant C \int_{\Omega} |\nabla u_k| dx + C$$
$$\leqslant C \int_{\Omega} |\nabla u_k|^{p(x)} dx + C \leqslant C \int_{\Omega} \frac{1}{p^+} |\nabla u_k|^{p(x)} dx + C \leqslant C \int_{\Omega} \frac{1}{p(x)} |\nabla u_k|^{p(x)} dx + C \leqslant C.$$

Together with the inequality

$$\int_{\Omega} |\nabla u_k|^{p(x)} dx \leqslant C,$$

we obtain $Q_{p(x)}(u_k) + Q_{p(x)}(\nabla u_k) \leqslant C$, which implies $\{u_k\}_{k=1}^{\infty}$ is uniformly bounded in $W^{1,p(x)}(\Omega)$. Meanwhile, $\int_{\Omega} (u_k - f)^2 dx \leqslant C$ results in the uniformly boundedness of $\{u_k\}_{k=1}^{\infty}$ in $L^2(\Omega)$. Since $W^{1,p(x)}(\Omega) \cap L^2(\Omega)$ is a reflexive Banach space, there exists a subsequence $\{u_{k_j}\}_{j=1}^{\infty} \subset \{u_k\}_{k=1}^{\infty}$, and $u \in W^{1,p(x)}(\Omega) \cap L^2(\Omega)$ such that

$$u_{k_j} \rightharpoonup u \text{ in } W^{1,p(x)}(\Omega) \cap L^2(\Omega).$$

Moreover, since $\{u_k\}_{k=1}^{\infty} \subset U_a$, we conclude that $u \in W^{1,p(x)} \cap L^{\infty}(\Omega)$. We assert next that, $u - w \in W_0^{1,p(x)} \cap L^{\infty}(\Omega)$. To see this, note that for $w \in W^{1,p(x)}(\Omega) \cap L^{\infty}(\Omega), u_k - w \in W_0^{1,p(x)}(\Omega) \cap L^{\infty}(\Omega)$. Since $W_0^{1,p(x)}(\Omega) \cap L^{\infty}(\Omega)$ is a closed, linear subspace of $W^{1,p(x)}(\Omega) \cap L^{\infty}(\Omega)$, it is weakly closed. Hence $u - w \in W_0^{1,p(x)}(\Omega) \cap L^{\infty}(\Omega)$. Then by Lemma 2.4,

$$E_{p(x)}(u) \leqslant \liminf_{j\to\infty} E_{p(x)}(u_{k_j}) = \mu.$$

Therefore, we conclude that u is a minimizer of $E_{p(x)}$. The uniqueness follows from the strictly convexity of $E_{p(x)}(u)$ in u.

In the proof of Theorem 2.2, we use $W^{1,1}$ – Sobolev–Poincaré inequality which is different from Theorem 2.1. In special case with no fidelity term in the energy (called $p(x)$-Laplacian Dirichlet problem), the existence of minimizer has been studied in [5,7].

Assume $w \in W^{1,p(x)}(\Omega) \cap L^{\infty}(\Omega)$, and $f - w \in W_0^{1,p(x)}(\Omega) \cap L^{\infty}(\Omega)$ which means f has fixed boundary value $f|_{\partial\Omega} = w|_{\partial\Omega}$. Let u be the minimizer of problem (2.1). We calculate the corresponding Euler–Lagrange equation.

Taking $\varphi \in W_0^{1,p(x)}(\Omega)$ as a test function, we have $u + \epsilon\varphi - w \in W_0^{1,p(x)}(\Omega)$ for each $\epsilon > 0$. Then

$$\begin{aligned}\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} E_{p(x)}(u+\epsilon\varphi) &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\left\{\int_\Omega \frac{1}{p(x)}|\nabla u+\epsilon\nabla\varphi|^{p(x)}dx+\frac{\lambda}{2}\int_\Omega(u+\epsilon\varphi-f)^2dx\right\}\\ &= \left.\left\{\int_\Omega |\nabla u+\epsilon\nabla\varphi|^{p(x)-1}\frac{\nabla u+\epsilon\nabla\varphi}{|\nabla u+\epsilon\nabla\varphi|}\nabla\varphi+\lambda(u+\epsilon\varphi-f)\varphi\,dx\right\}\right|_{\epsilon=0}\\ &= \int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla\varphi+\lambda(u-f)\varphi\,dx=0.\end{aligned}$$

We have that the minimizer of problem (2.1) satisfies the following equation with Dirichlet boundary condition:

$$\begin{cases}\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)-\lambda(u-f)=0, & x\in\Omega,\\ u=w. & x\in\partial\Omega.\end{cases}$$

Similarly, the minimizer of problem (1.4) satisfies the following equation with Neumann boundary condition

$$\begin{cases}\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)-\lambda(u-f)=0, & x\in\Omega,\\ \frac{\partial u}{\partial N}=0, & x\in\partial\Omega,\end{cases}$$

where N denotes the unit outward normal of $\partial\Omega$.

In [13], entropy solution for the $p(x)$-Laplace equation without fidelity term was studied. In a different perspective, we study the associated flow corresponding to the Euler–Lagrange equation in this paper. In the following, we only study the problem with Neumann boundary condition. Remark that Neumann or periodic boundary conditions are more natural for image processing applications. However, the Dirichlet boundary conditions are a bit more interesting mathematically and all of the same proofs hold (in a simplified manner) for the Neumann conditions. □

3. The associated heat flow to problem (1.4)

Using the steepest descent method, the associated heat flow to problem (1.4) is given by

$$u_t=\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)-\lambda(u-f),\quad (x,t)\in\Omega^T, \tag{3.1}$$

$$\frac{\partial u}{\partial N}=0,\quad (x,t)\in\partial\Omega^T, \tag{3.2}$$

$$u(0)=f,\quad (x,t)\in\Omega\times\{t=0\}. \tag{3.3}$$

Firstly, we derive another definition of weak solution of problem (3.1)–(3.3). Denote

$$F(\nabla u,u,x)=\frac{1}{p(x)}|\nabla u|^{p(x)}+\frac{\lambda}{2}(u-f)^2.$$

Then (3.1) is equivalent to $u_t=-F'(\nabla u,u,x)$, where $F'(\nabla u,u,x)$ denotes the Gateaux derivative of F about u.

Suppose u be a classical solution of (3.1)–(3.3). For each $v\in L^2(0,T;W^{1,p(x)}(\Omega)\cap L^2(\Omega))$, multiplying (3.1) by $v-u$, and then integrating over Ω, we have that

$$\int_\Omega u_t(v-u)dx=\int_\Omega -F'(\nabla u,u,x)(v-u)dx.$$

From the convexity of $F(\nabla u,u,x)$, we deduce that

$$\int_\Omega u_t(v-u)dx+E_{p(x)}(v)\geqslant E_{p(x)}(u). \tag{3.4}$$

Integrating over $[0,s]$ for any $s\in[0,T]$ yields

$$\int_0^s\int_\Omega u_t(v-u)dx\,dt+\int_0^s E_{p(x)}(v)dt\geqslant\int_0^s E_{p(x)}(u)dt. \tag{3.5}$$

On the other hand, if (3.5) holds, setting $v=u+\epsilon\varphi$ in (3.5) with $\varphi\in C_0^\infty(\Omega)$, we obtain

$$\int_0^s\int_\Omega u_t\epsilon\varphi\,dx\,dt+\int_0^s E_{p(x)}(u+\epsilon\varphi)dt\geqslant\int_0^s E_{p(x)}(u)dt,$$

which implies $\int_0^s\int_\Omega u_t\epsilon\varphi\,dx\,dt+\int_0^s E_{p(x)}(u+\epsilon\varphi)dt$ attains its minimum at $\epsilon=0$. Hence

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\left(\int_0^s\int_\Omega u_t\epsilon\varphi\,dx\,dt+\int_0^s E_{p(x)}(u+\epsilon\varphi)dt\right)=0,$$

that is,

$$\int_0^s \int_\Omega \dot{u}\varphi \, dx \, dt + \int_0^s \int_\Omega F'(\nabla u, u, x)\varphi \, dx \, dt = 0.$$

Since φ is arbitrary, $\dot{u} + F'(\nabla u, u, x) = 0$. That is to say, if u satisfies (3.5), then u is a weak solution of (3.1) in the sense of distribution. This motivates us to give the following definition.

Definition. A function $u \in L^2(0, T; W^{1,p(x)}(\Omega) \cap L^2(\Omega))$, with $\dot{u} \in L^2(\Omega^T)$ is called a weak solution of Eqs. (3.1)–(3.3) if $u(0) = f$, and for all $v \in L^2(0, T; W^{1,p(x)}(\Omega) \cap L^2(\Omega))$, for all $s \in [0, T]$, (3.5) holds.

Let $\varphi(\xi) = \frac{1}{p(x)}|\xi|^{p(x)}$, then the derivative is $\varphi_r(\xi) = |\xi|^{p(x)-2}\xi$. Setting

$$\varphi^\epsilon(\xi) = \frac{1}{p(x)}\left(\sqrt{|\xi|^2 + \epsilon^2}\right)^{p(x)}$$

with $0 < \epsilon < 1$. Then

$$\varphi_r^\epsilon(\xi) = \frac{\xi}{\left(\sqrt{|\xi|^2 + \epsilon^2}\right)^{2-p(x)}}.$$

It is easy to see that $\varphi^\epsilon(\xi)$ is convex in ξ and $\varphi^\epsilon \to \varphi$ as $\epsilon \to 0$.

To prove the existence of solution to (3.1)–(3.3), we first discuss the solution of the approximated problem

$$u_t = \epsilon\Delta u + \operatorname{div}(\varphi_r^\epsilon(\nabla u)) - \lambda(u - f_\delta), \quad (x, t) \in \Omega^T, \tag{3.6}$$

$$\frac{\partial u}{\partial N} = 0, \quad (x, t) \in \partial\Omega^T, \tag{3.7}$$

$$u(0) = f_\delta, \quad (x, t) \in \Omega \times \{t = 0\}, \tag{3.8}$$

where $f_\delta \in C^\infty(\overline{\Omega})$ has the following properties:

$$f_\delta \to f \text{ in } L^2(\Omega), \quad \|f_\delta\|_{L^\infty(\Omega)} \leqslant \|f\|_{L^\infty(\Omega)}, \quad \varphi(\nabla f_\delta) \leqslant \varphi(\nabla f). \tag{3.9}$$

Since $1 < p(x) \leqslant 2$, for $\varphi(s) = \frac{1}{p(x)}s^{p(x)}$, we have $\varphi'(s) = \frac{1}{p(x)}p(x)s^{p(x)-1} = s^{p(x)-1} > 0 (s > 0)$. Hence φ is monotonically increasing function in s $(s > 0)$. The convexity of φ yields

$$\begin{aligned}\varphi^\epsilon(\nabla f_\delta) &= \frac{1}{p(x)}\left(\sqrt{|\nabla f_\delta|^2 + \epsilon^2}\right)^{p(x)} \leqslant \frac{1}{p(x)}(|\nabla f_\delta| + \epsilon)^{p(x)} \leqslant \frac{1}{p(x)}2^{p(x)-1}(|\nabla f_\delta|^{p(x)} + \epsilon^{p(x)}) \\ &\leqslant 2^{p(x)-1}\left(\varphi(\nabla f_\delta) + \frac{1}{p(x)}\epsilon^{p(x)}\right) \leqslant 2(\varphi(\nabla f) + \epsilon).\end{aligned} \tag{3.10}$$

The existence of f_δ can be proved by the standard argument as in [10].

Lemma 3.1. *The problem (3.6)–(3.8) has a unique weak solution u_δ^ϵ, with $u_\delta^\epsilon \in L^\infty(0, T; W^{1,p(x)}(\Omega) \cap L^2(\Omega))$ and $\dot{u}_\delta^\epsilon \in L^2(0, T; L^2(\Omega))$ such that*

$$\int_0^\infty \int_\Omega |\dot{u}_\delta^\epsilon|^2 dx\,dt + \sup_{t>0}\left\{\int_\Omega \frac{\epsilon}{2}|\nabla u_\delta^\epsilon|^2 + \varphi^\epsilon(\nabla u_\delta^\epsilon) + \frac{\lambda}{2}(u_\delta^\epsilon - f_\delta)^2 dx\right\} \leqslant 2\left\{\int_\Omega \frac{\epsilon}{2}|\nabla f_\delta|^2 dx + \varphi(\nabla f)dx + 1\right\}. \tag{3.11}$$

Proof. (3.6)–(3.8) is quasilinear parabolic equations of divergence type. Since it satisfies all necessary conditions which can be verified by directly calculation, (3.6)–(3.8) has a unique weak solution u_δ^ϵ [9]. Then u_δ^ϵ satisfies (3.6). Multiplying (3.6) by $\dot{u}_\delta^\epsilon$ and integrating on Ω, we get

$$\int_\Omega |\dot{u}_\delta^\epsilon|^2 = \int_\Omega \epsilon\dot{u}_\delta^\epsilon\Delta u_\delta^\epsilon dx + \int_\Omega \dot{u}_\delta^\epsilon \operatorname{div}(\varphi_r^\epsilon(\nabla u_\delta^\epsilon))dx - \lambda\int_\Omega \dot{u}_\delta^\epsilon(u_\delta^\epsilon - f_\delta)dx.$$

Then

$$\int_\Omega |\dot{u}_\delta^\epsilon|^2 dx + \frac{d}{dt}\left\{\int_\Omega \frac{\epsilon}{2}|\nabla u_\delta^\epsilon|^2 + \varphi^\epsilon(\nabla u_\delta^\epsilon) + \frac{\lambda}{2}(u_\delta^\epsilon - f_\delta)^2 dx\right\} = 0.$$

Integrating the above formula on $(0, t)$,

$$\int_0^s \int_\Omega |\dot{u}_\delta^\epsilon|^2 dx\,dt + \left\{\int_\Omega \frac{\epsilon}{2}|\nabla u_\delta^\epsilon|^2 + \varphi^\epsilon(\nabla u_\delta^\epsilon) + \frac{\lambda}{2}(u_\delta^\epsilon - f_\delta)^2 dx\right\} = \left\{\int_\Omega \frac{\epsilon}{2}|\nabla f_\delta|^2 + \varphi^\epsilon(\nabla f_\delta) + \frac{\lambda}{2}(f_\delta - f_\delta)^2 dx\right\}.$$

Therefore,

$$\int_0^{\infty}\int_\Omega |\dot{u}_\delta^\epsilon|^2 dx\,dt + \sup_{t>0}\left\{\int_\Omega \frac{\epsilon}{2}|\nabla u_\delta^\epsilon|^2 + \varphi^\epsilon(\nabla u_\delta^\epsilon) + \frac{\lambda}{2}(u_\delta^\epsilon - f_\delta)^2 dx\right\} \leqslant 2\left\{\int_\Omega \frac{\epsilon}{2}|\nabla f_\delta|^2 + \varphi^\epsilon(\nabla f_\delta)dx\right\}.$$

Since $0<\epsilon<1$, we get the conclusion. □

Lemma 3.2. *Let $f \in W^{1,p(x)} \cap L^\infty(\Omega)$, and u_δ^ϵ be the weak solution of the problem (3.6)–(3.8). Then*

$$\|u_\delta^\epsilon\|_{L^\infty(\Omega^T)} \leqslant \|f\|_{L^\infty(\Omega)}. \tag{3.12}$$

Proof. Let G be a truncation function of class C^1 such that $G(t)=0$ on $(-\infty,0]$, and G is strictly increasing in $[0,+\infty)$, and $G' \leqslant M$ where M is a constant. Let $k=\|f\|_{L^\infty(\Omega)}$ and set $v = G(u_\delta^\epsilon - k)$. Since $u_\delta^\epsilon \in W^{1,p(x)}(\Omega)\cap L^2(\Omega)$, by the chain rule we get $v \in W^{1,p(x)}(\Omega)\cap L^2(\Omega)$, and $\nabla v = G'(u_\delta^\epsilon - k)\nabla u_\delta^\epsilon$. Multiplying (3.6) by v and integrating over Ω yields

$$0=\int_\Omega \dot{u}_\delta^\epsilon G(u_\delta^\epsilon - k)dx + \epsilon\int_\Omega |\nabla u_\delta^\epsilon|^2 G'(u_\delta^\epsilon - k)dx + \int_\Omega \varphi_r^\epsilon(\nabla u_\delta^\epsilon)\nabla u_\delta^\epsilon G'(u_\delta^\epsilon - k)dx + \lambda\int_\Omega (u_\delta^\epsilon - f_\delta)G(u_\delta^\epsilon - k)dx. \tag{3.13}$$

By the definition of φ^ϵ, $\int_\Omega \varphi_r^\epsilon(\nabla u_\delta^\epsilon)\nabla u_\delta^\epsilon G'(u_\delta^\epsilon - k)dx \geqslant 0$. It is obvious that $\epsilon\int_\Omega |\nabla u_\delta^\epsilon|^2 G'(u_\delta^\epsilon - k)dx \geqslant 0$. If $\int_\Omega (u_\delta^\epsilon - f_\delta) G(u_\delta^\epsilon - k)dx \leqslant 0$, then we get $u_\delta^\epsilon \leqslant \|f_\delta\|_{L^\infty(\Omega)} \leqslant \|f\|_{L^\infty(\Omega)} = k$. Otherwise, we have $\int_\Omega (u_\delta^\epsilon - f_\delta)G(u_\delta^\epsilon - k)dx \geqslant 0$. Hence (3.13) yields

$$\int_\Omega \dot{u}_\delta^\epsilon G(u_\delta^\epsilon - k)dx \leqslant 0.$$

Since $0 \leqslant G' \leqslant M$ we deduce that

$$\frac{d}{dt}\int_\Omega (G(u_\delta^\epsilon - k))^2 dx \leqslant 0.$$

Therefore $\int_\Omega (G(u_\delta^\epsilon - k))^2 dx$ is monotonically decreasing function about t and then

$$\int_\Omega (G(u_\delta^\epsilon - k))^2 dx \leqslant \int_\Omega (G(u_\delta^\epsilon - k))^2 dx|_{t=0} = \int_\Omega (G(f_\delta - k))^2 dx = 0.$$

So we have proved that $u_\delta^\epsilon \leqslant k$. Similarly, $u_\delta^\epsilon \geqslant -k$ can be proved. □

Theorem 3.1 (existence and uniqueness). *Suppose $f \in W^{1,p(x)}(\Omega)\cap L^\infty(\Omega)$. Then (3.1)–(3.3) has a unique weak solution $u \in L^\infty(0,T;W^{1,p(x)}(\Omega)\cap L^\infty(\Omega))$, with $\dot{u} \in L^2(\Omega^T)$.*

Proof. First we fix $\delta>0$ and pass to the limit $\epsilon \to 0$. Let $\{u_\delta^\epsilon\}$ be the sequence of solution to (3.6)–(3.8). By (3.11) and (3.12), we get that $\{u_\delta^\epsilon\}$ has uniformly bounded $L^\infty(\Omega^\infty)$ norm about ϵ, and $\{\dot{u}_\delta^\epsilon\}$ has uniformly bounded $L^2(\Omega^\infty)$ norm. Then there exists a subsequence, also denoted by $\{u_\delta^\epsilon\}$, and a function $u_\delta \in L^\infty(\Omega^\infty)$, such that as $\epsilon \to 0$,

$$u_\delta^\epsilon \rightharpoonup u_\delta \text{ weakly } * \text{ in } L^\infty(\Omega^\infty), \tag{3.14}$$

$$\dot{u}_\delta^\epsilon \rightharpoonup w \text{ weakly in } L^2(\Omega^\infty). \tag{3.15}$$

The same argument used in the proof of Lemma 3.1 [16] gives us that $\dot{u}_\delta = w, u_\delta(0) = f_\delta$. Then we have $\dot{u}_\delta \in L^2(\Omega^\infty)$.

Moreover, for all $\phi \in L^2(\Omega)$,

$$\int_\Omega (u_\delta^\epsilon(\cdot,t) - f_\delta)\phi(x)dx = \int_0^t\int_\Omega \dot{u}_\delta^\epsilon(x,s)1_{[0,t]}(s)\phi(x)dx\,ds \to \int_0^t\int_\Omega \dot{u}_\delta(x,s)1_{[0,t]}(s)\phi(x)dx\,dt = \int_\Omega (u_\delta(\cdot,t) - f_\delta)\phi(x)dx \quad (\epsilon \to 0),$$

which implies that

$$u_\delta^\epsilon(\cdot,t) \rightharpoonup u_\delta(\cdot,t) \text{ weakly in } L^2(\Omega).$$

From (3.11), for each $t>0$, $\{u_\delta^\epsilon(\cdot,t)\}$ is a uniformly bounded sequence in $W^{1,1}(\Omega)$. Then there exists a subsequence, also denoted by $\{u_\delta^\epsilon(\cdot,t)\}$, such that

$$u_\delta^\epsilon(\cdot,t) \to u_\delta(\cdot,t) \text{ strongly in } L^1(\Omega). \tag{3.16}$$

From (3.12), (3.14) and (3.16), we obtain

$$\int_\Omega |u_\delta^\epsilon(\cdot,t) - u_\delta(\cdot,t)|^2 dx \leqslant \|u_\delta^\epsilon(\cdot,t) - u_\delta(\cdot,t)\|_{L^\infty(\Omega)}\int_\Omega |u_\delta^\epsilon(\cdot,t) - u_\delta(\cdot,t)|dx \leqslant C\|f\|_{L^\infty(\Omega)}\int_\Omega |u_\delta^\epsilon(\cdot,t) - u_\delta(\cdot,t)|dx\,dt \to 0 \qquad (as\ \epsilon \to 0).$$

Therefore,

$$u_\delta^\epsilon(\cdot,t) \to u_\delta(\cdot,t) \text{ strongly in } L^2(\Omega). \tag{3.17}$$

For all $v \in L^2((0,\infty); H^1(\Omega))$, multiplying (3.6) (where u is replaced by u_δ^ϵ) by $(v - u_\delta^\epsilon)$ and using the convexity of φ^ϵ, we get

$$\int_0^s \int_\Omega \dot{u}_\delta^\epsilon (v - u_\delta^\epsilon) + \frac{\epsilon}{2}|\nabla v|^2 + \varphi^\epsilon(\nabla v) + \frac{\lambda}{2}(v - f_\delta)^2 dx\,dt \geqslant \int_0^s \int_\Omega \frac{\epsilon}{2}|\nabla u_\delta^\epsilon|^2 + \varphi^\epsilon(\nabla u_\delta^\epsilon) + \frac{\lambda}{2}(u_\delta^\epsilon - f_\delta)^2 dx\,dt.$$

From (3.15), (3.17) and the lower semi-continuity of φ^ϵ we obtain

$$\int_0^s \int_\Omega \dot{u}_\delta^\epsilon (v - u_\delta^\epsilon) + \varphi^\epsilon(\nabla v) + \frac{\lambda}{2}(v - f_\delta)^2 dx\,dt \geqslant \liminf_{\epsilon\to 0} \int_0^s \int_\Omega \varphi^\epsilon(\nabla u_\delta^\epsilon) + \frac{\lambda}{2}(u_\delta^\epsilon - f_\delta)^2 dx\,dt.$$

Letting $\epsilon \to 0$, we obtain that

$$\int_0^s \int_\Omega \dot{u}_\delta (v - u_\delta) + \varphi(\nabla v) + \frac{\lambda}{2}(v - f_\delta)^2 dx\,dt \geqslant \int_0^s \int_\Omega \varphi(\nabla u_\delta) + \frac{\lambda}{2}(u_\delta - f_\delta)^2 dx\,dt \tag{3.18}$$

holds for all $v \in L^2(0,\infty; H^1(\Omega))$. By approximation, (3.18) still holds for any $v \in L^2((0,\infty); W^{1,p(x)}(\Omega) \cap L^2(\Omega))$.

It remains to pass to the limit as $\delta \to 0$. In (3.11) we let $\epsilon \to 0$ to get that

$$\int_0^\infty \int_\Omega |\dot{u}_\delta|^2 dx\,dt + \sup_{t>0} \int_\Omega \left\{ \varphi(\nabla u_\delta) + \frac{\lambda}{2}(u_\delta - f_\delta)^2 dx \right\} \leqslant C,$$

where C depends on f. Therefore, $\{u_\delta\}$ is uniformly bounded in $L^\infty(0,\infty; W^{1,p(x)}(\Omega) \cap L^2(\Omega))$, and then uniformly bounded in $W^{1,1}(\Omega)$. We also have $\dot{u}^\delta$ is uniformly bounded in $L^2(\Omega^\infty)$. Moreover, letting $\epsilon \to 0$ in (3.12) yields

$$\|u_\delta\|_{L^\infty(\Omega^\infty)} \leqslant \|f\|_{L^\infty(\Omega)}.$$

Hence $\{u_\delta\}$ is uniformly bounded in $L^\infty(\Omega^\infty)$. By the same argument used in (3.14), (3.15) and (3.17), there exists a subsequence, also denoted by $\{u_\delta\}$ and a function $u \in L^\infty((0,\infty); W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)), \dot{u} \in L^2(\Omega^\infty)$ such that as $\delta \to 0$,

$$u_\delta \rightharpoonup u \text{ weakly} * \text{in } L^\infty(\Omega^\infty), \tag{3.19}$$

$$\dot{u}_\delta \rightharpoonup \dot{u} \text{ weakly in } L^2(\Omega^\infty), \tag{3.20}$$

$$u_\delta(\cdot,t) \to u(\cdot,t) \text{ strongly in } L^2(\Omega) \text{ and uniformly for } t. \tag{3.21}$$

Using the lower semi-continuity of φ and (3.19)–(3.21), and letting $\delta \to 0$ in (3.18), we conclude that for all $v \in W^{1,p(x)}(\Omega) \cap L^2(\Omega)$

$$\int_0^s \int_\Omega u_t(v - u)dx\,dt + \int_0^s E_{p(x)}(v)dt \geqslant \int_0^s E_{p(x)}(v)dt.$$

By definition, u is the weak solution of problem (3.1)–(3.3).

Uniqueness follows directly from the following stability theorem by letting $f_1 = f_2$. □

Theorem 3.2 (stability). *Assume u_1 and u_2 are both weak solutions of (3.1)–(3.3) with initial values $f_1, f_2 \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$. Then for any $t > 0$,*

$$\|u_1 - u_2\|_{L^\infty(\Omega)} \leqslant \|f_1 - f_2\|_{L^\infty(\Omega)}.$$

Proof. Set $k = \|f_1 - f_2\|_{L^\infty(\Omega)}$. Define

$$\begin{cases} v = u_1 - (u_1 - u_2 - k)_+ \\ w = u_2 + (u_1 - u_2 - k)_+ \end{cases},$$

where

$$(u_1 - u_2 - k)_+ = \begin{cases} u_1 - u_2 - k, & \text{if } u_1 - u_2 - k \geqslant 0, \\ 0, & \text{otherwise}. \end{cases}$$

Then

$$\nabla v = \begin{cases} \nabla u_1, & u_1 - u_2 \leqslant k, \\ \nabla u_2, & u_1 - u_2 \geqslant k, \end{cases} \quad \nabla w = \begin{cases} \nabla u_2, & u_1 - u_2 \leqslant k, \\ \nabla u_1. & u_1 - u_2 \geqslant k. \end{cases}$$

By the definition of weak solution of (3.1)–(3.3), for all $t > 0$, we have

$$\int_{\Omega} \dot{u}_1(v-u_1)+\varphi(\nabla v)+\frac{\lambda}{2}(v-f_1)^2dx\,dt \geqslant \int_{\Omega}\varphi(\nabla u_1)+\frac{\lambda}{2}(u_1-f_1)^2dx\,dt,$$
$$\int_{\Omega} \dot{u}_2(w-u_2)+\varphi(\nabla w)+\frac{\lambda}{2}(w-f_2)^2dx\,dt \geqslant \int_{\Omega}\varphi(\nabla u_2)+\frac{\lambda}{2}(u_2-f_2)^2dx\,dt.$$

Taking summation yields

$$\int_{\Omega} \dot{u}_1(v-u_1)+\dot{u}_2(w-u_2)+\varphi(\nabla v)+\varphi(\nabla w)+\frac{\lambda}{2}(v-f_1)^2+\frac{\lambda}{2}(w-f_2)^2dx\,dt$$
$$\geqslant \int_{\Omega}\varphi(\nabla u_1)+\varphi(\nabla u_2)+\frac{\lambda}{2}(u_1-f_1)^2+\frac{\lambda}{2}(u_2-f_2)^2dx\,dt.$$

By the definition of v and w, it is clear that

$$\varphi(\nabla v)+\varphi(\nabla w)=\varphi(\nabla u_1)+\varphi(\nabla u_2),$$

and

$$\begin{aligned}\int_{\Omega}(u_1-f_1)^2+(u_2-f_2)^2-(v-f_1)^2-(w-f_2)^2dx &= \int_{\Omega}(u_1-f_1)(u_1+v-2f_1)+(u_2-w)(u_2+w-2f_2)dx\\ &= \int_{\Omega}(u_1-u_2-k)_+(2u_1-2f_1-2u_2+2f_2-2(u_1-u_2-k)_+)dx\\ &= \int_{\Omega}2(u_1-u_2-k)_+((u_1-u_2-k)-(u_1-u_2-k)_+-(f_1-f_2-k))dx.\end{aligned}$$

By the definition of k, we have $f_1-f_2-k\leqslant 0$. If $(u_1-u_2-k)_+=0$, then

$$\int_{\Omega}(u_1-u_2-k)_+((u_1-u_2-k)-(u_1-u_2-k)_+-(f_1-f_2-k))dx=0.$$

If $(u_1-u_2-k)_+>0$, then

$$\int_{\Omega}(u_1-u_2-k)_+((u_1-u_2-k)-(u_1-u_2-k)_+-(f_1-f_2-k))dx$$
$$\geqslant \int_{\Omega}(u_1-u_2-k)_+((u_1-u_2-k)-(u_1-u_2-k))dx=0.$$

So we get

$$\int_{\Omega}((v-f_1)^2+(w-f_2)^2)dx\leqslant\int_{\Omega}((u_1-f_1)^2+(u_2-f_2)^2)dx.$$

Therefore,

$$\int_{\Omega}(\dot{u}_1(v-u_1)+\dot{u}_2(w-u_2))dx\geqslant 0.$$

By the definition of v and w, we get

$$\int_{\Omega}(\dot{u}_1-\dot{u}_2)(u_1-u_2-k)_+dx\leqslant 0,$$

that is,

$$\frac{d}{dt}\int_{\Omega}|(u_1-u_2-k)_+|^2dx\leqslant 0.$$

So $\int_{\Omega}|(u_1-u_2-k)_+|^2dx$ is monotonically decreasing function of t, then

$$\int_{\Omega}|(u_1-u_2-k)_+|^2dx\leqslant\int_{\Omega}|(f_1-f_2-k)_+|^2dx(=0).$$

Therefore, $u_1-u_2\leqslant k$. Similarly we can prove $u_1-u_2\geqslant -k$ □

Theorem 3.3 (long time behavior). *Let $f\in W^{1,p(x)}(\Omega)\cap L^{\infty}(\Omega)$. Then as $t\to\infty$, the weak solution $u(x,t)$ to (3.1)–(3.3) converges strongly to the solution of the minimization problem (1.4) in $L^2(\Omega)$.*

Proof. By the definition of weak solution (3.5), for all $s>0$ and for all $v(x)\in W^{1,p(x)}(\Omega)\cap L^2(\Omega)$, we have

$$\int_0^s\int_{\Omega}\dot{u}(x,t)(v(x)-u(x,t))dx\,dt+\int_0^s E_{p(x)}(v(x))dt\geqslant\int_0^s E_{p(x)}(u(x,t))dt,$$

that is,

$$\int_{\Omega}(u(x,s)-f(x))v(x)dx-\frac{1}{2}\int_{\Omega}(u^2(x,s)-f^2(x))dx+s\int_{\Omega}\varphi(\nabla v(x))dx+s\frac{\lambda}{2}\int_{\Omega}(v(x)-f(x))^2dx$$
$$\geqslant \int_0^s\int_{\Omega}\varphi(\nabla u)dx\,dt+\frac{\lambda}{2}\int_0^s\int_{\Omega}(u-f)^2dx\,dt. \tag{3.22}$$

Define

$$w(x,s)=\frac{1}{s}\int_0^s u(x,t)dt.$$

Since $u\in L^{\infty}(0,\infty;W^{1,p(x)}(\Omega)\cap L^{\infty}(\Omega))$ for any $s>0$, we have $w(\cdot,s)$ is uniformly bounded in $W^{1,p(x)}(\Omega)$ and $L^{\infty}(\Omega)$. Then there exists a subsequence, also denoted by $\{w(x,s)\}$, and a function $\tilde{u}\in W^{1,p(x)}(\Omega)\cap L^{\infty}(\Omega)$, such that

$$w(x,s)\rightharpoonup\tilde{u}\ \text{in}\ W^{1,p(x)}(\Omega),$$
$$w(x,s)\to\tilde{u}\ \text{in}\ L^2(\Omega).$$

Dividing (3.22) by s, then letting $s\to\infty$, we obtain

$$E_{p(x)}(v)\geqslant E_{p(x)}(\tilde{u})$$

for all $v(x)\in W^{1,p(x)}(\Omega)\cap L^2(\Omega)$. Hence, $\tilde{u}$ is the solution to problem (1.4). □

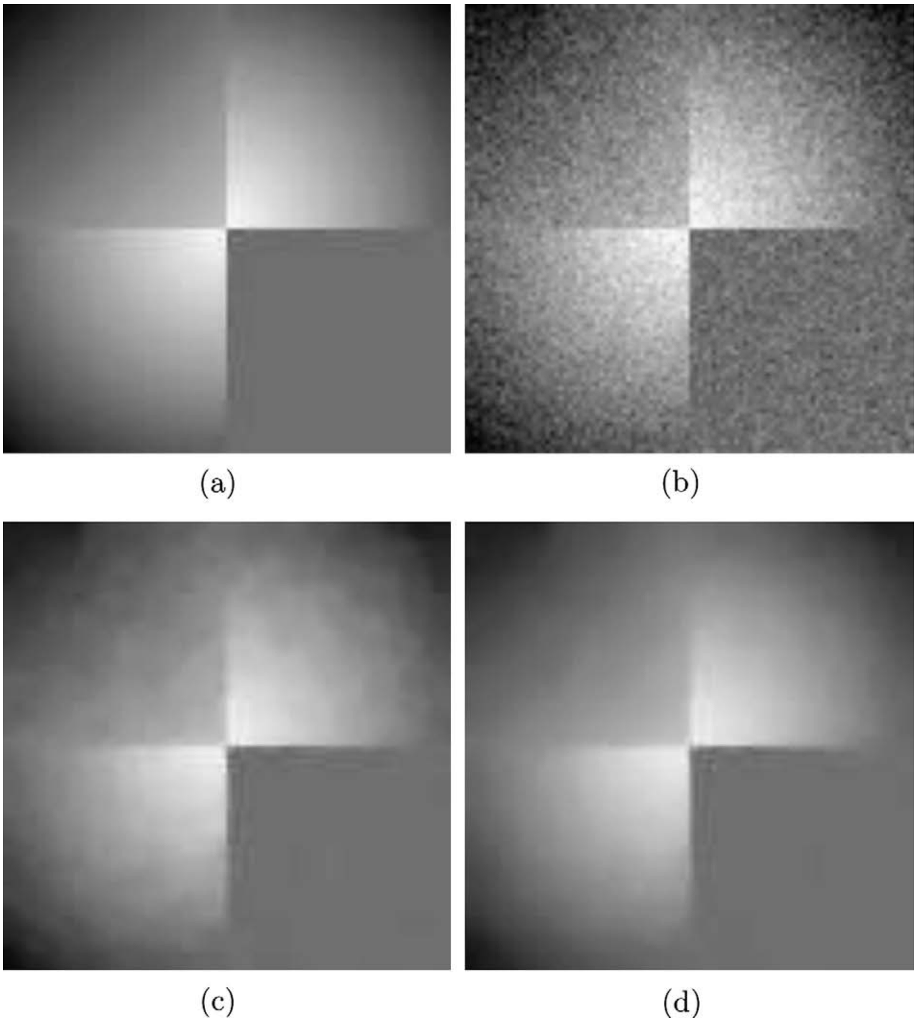

Fig. 1. Comparison of the proposed model and the ROF model. (a) The true image; (b) the noisy image; (c) the restoration result by the ROF model; and (d) the restoration result by the proposed model.

4. Numerical results

We consider dimension $n = 2$. Suppose the image size is $N \times N$. Set τ be the time step and $h = 1$ be the space step. Let $x_i = ih,\ y_j = jh,\ i,j = 0,1,\ldots,N,\ t_n = n\tau, n = 0,1,\ldots,\ u^n_{i,j} = u(x_i,y_j,t_n),\ u^0_{ij} = f(x_i,y_j)$. Define

$$(D^{\pm}_x u)_{i,j} = \pm[u_{i\pm1,j} - u_{i,j}], (D^{\pm}_y u)_{i,j} = \pm[u_{i,j\pm1} - u_{i,j}],$$
$$|(D_x u)_{i,j}| = \sqrt{(D^+_x(u_{i,j}))^2 + (m[D^+_y(u_{i,j}), D^-_y(u_{i,j})])^2 + 0.001},$$
$$|(D_y u)_{i,j}| = \sqrt{(D^+_y(u_{i,j}))^2 + (m[D^+_x(u_{i,j}), D^-_x(u_{i,j})])^2 + 0.001},$$

where $m[a,b] = \left(\frac{\text{sign } a + \text{sign } b}{2}\right) \cdot \min(|a|,|b|)$. Then the finite difference scheme of the heat flow (3.1)–(3.3) is given by

$$u^{k+1} = u^k + \tau\left(D^-_x\left(\frac{D^+_x u^k}{|D_x u^k|^{1-g}}\right) + D^-_y\left(\frac{D^+_y u^k}{|D_y u^k|^{1-g}}\right) - \lambda(u^k - f)\right),$$
$$u^0 = f,$$

where the subscripts i,j are omitted for simplicity. Remark that the Neumann boundary condition (3.2) is implemented by extend the image matrix symmetrically. To illustrate that our model has the advantage of reducing the staircasing effect while preserving edges, we run the ROF model as comparison. The numerical scheme of the ROF model is according to [12].

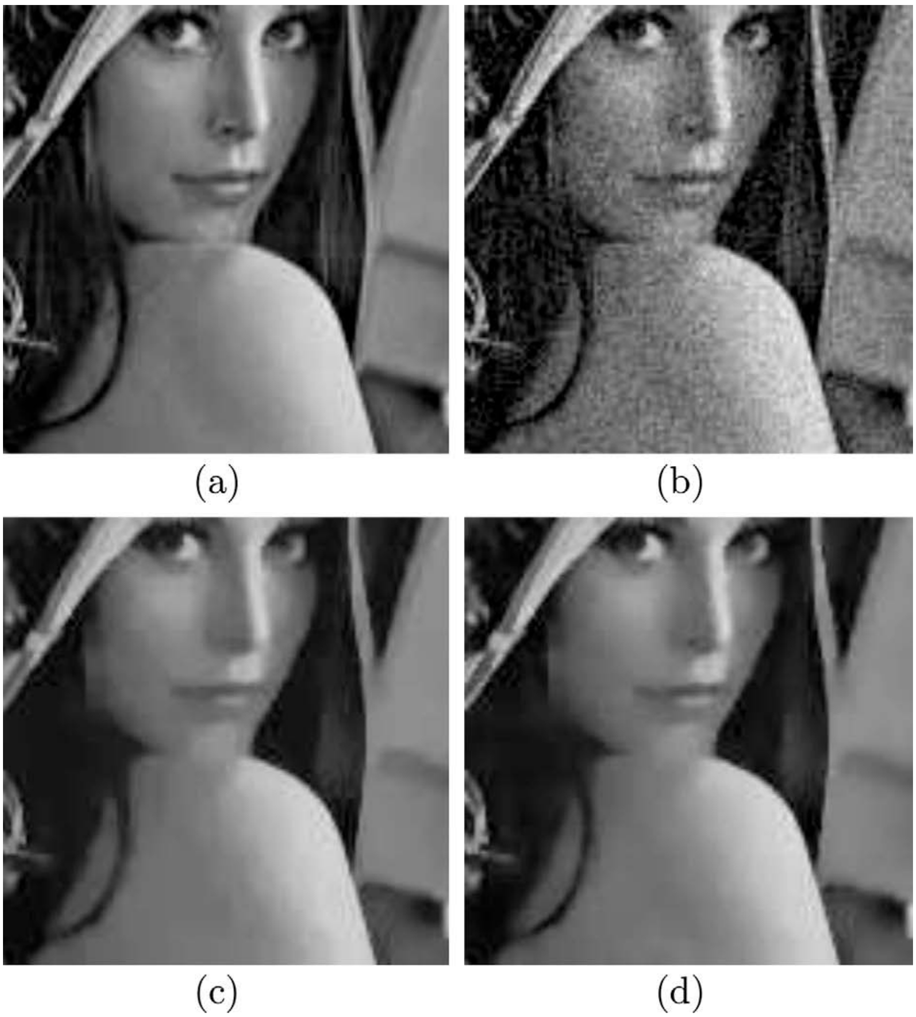

Fig. 2. Comparison of the proposed model and the ROF model. (a) A part of Lena image; (b) the noisy image; (c) the restoration result by the ROF model; and (d) the restoration result by the proposed model.

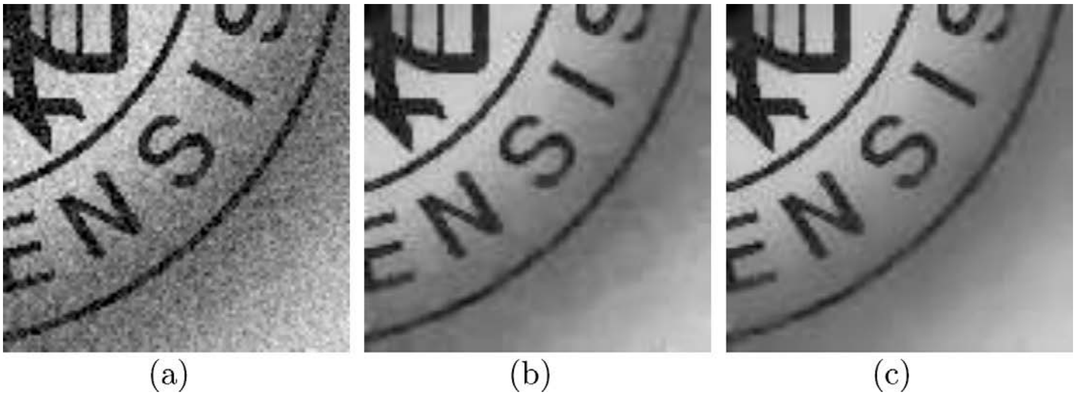

Fig. 3. Comparison of the proposed model and the ROF model. (a) The noisy image; (b) the restoration result by the ROF model; and (c) the restoration result by the proposed model.

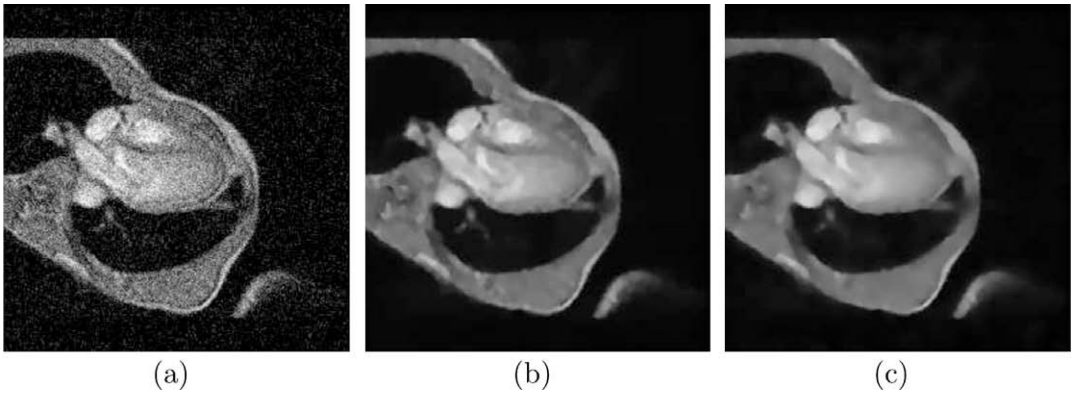

Fig. 4. Comparison of the proposed model and the ROF model. (a) Noisy MRI image of a heart; (b) the restoration result by the ROF model; and (c) the restoration result by the proposed model.

In all the experiments in this paper, the time step is set as 0.05, the fidelity coefficient λ is set as 0.01, $k = 0.005$, and $\sigma = 0.5$. The stopping criterion of both the ROF model and our model is the relative difference of the restored image should satisfy the following inequality:

$$\frac{\|u^{k+1} - u^k\|_2}{\|u^{k+1}\|_2} < 10^{-4}.$$

In Fig. 1, a typical piecewise smooth image is tested. Fig. 1(b) is the noisy version of Fig. 1(a). The restoration results by the ROF model and the proposed model are showed in Fig. 1(c) and (d), respectively. We can see in both results that the edges in the centerlines are preserved. However, in Fig. 1(c) the staircasing effect is obvious in the smooth regions, while in Fig. 1(d) the staircasing effect is successfully reduced.

A part of Lena image is tested in Fig. 2. Fig. 2(b) is the noisy version of Fig. 2(a). Figs. 2(c) and (d) show the restoration results by the ROF model and the proposed model, respectively. We can see that the proposed model recovers sharp edges as effectively as the ROF model. Meanwhile, in the smooth regions such as the shoulder, the staircasing effect can be seen in Fig. 2(c), while in Fig. 2(d) almost no staircasing effect occurs in these regions such that it seems more natural.

In Fig. 3, we test a character image with smooth background. Fig. 3(a) shows the noisy image. Figs. 3(b) and (c) show the restoration results by the ROF model and the proposed model, respectively. We observe that in both results the edges of the characters are preserved. Meanwhile, almost no staircasing effect appears when processed by the proposed model.

We test a medical image in Fig. 4. An MRI image of a heart with noise is showed in Fig. 4(a). Figs. 4(b) and (c) show the restoration results by the ROF model and the proposed model, respectively. The staircasing effect is obvious on the surface of the organ in Fig. 4(b), in contrast, almost no staircasing effect occurs in Fig. 4(c) where the organ surface is smooth.

5. Conclusion

In this paper, we have studied a variational exponent $(1 < p(x) \leqslant 2)$ functional to recover images based on the models (1.2) and (1.3). The significant difference between our model and (1.3) is that in our model (1.4) $p(x)$ can approximate 1 (but larger than 1) while in (1.3) $p(x)$ will be equal to 1 in regions with large gradient. However, theoretically, the two models are discussed in different spaces. (1.3) is studied in BV space while (1.4) is studied in variable exponent Sobolev space $W^{1,p(x)}$.

The case that includes $p(x) = 1$ is interesting. Some lemmas in Section 3 no longer hold any more. If $1 \leqslant p(x) \leqslant 2$, other kind of variable exponent space (not $W^{1,p(x)}$) should be introduced. This will be the future work.

Acknowledgements

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