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Poisson Noise Removal with Total Variation Regularization and Local Fidelity*

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Abstract — In this paper, we propose two methods to choose the fidelity parameter in total variation based denoising model for Poisson noise. Firstly, we derive a scheme to choose the scalar parameter automatically. Secondly, we propose to use a local fidelity term with spatial-varying parameters which automatically controls the extent of denoising according to image contents. Experiments with simulated data demonstrate that the proposed algorithms are effective.

Key words — Poisson noise, Local fidelity term, Total variation, Texture.

I. Introduction

Image denoising is one of the fundamental problems in image processing. It is aimed to smooth a noisy image without losing significant features such as edges and textures. Variational denoising methods have become popular in recent years. Most of the literatures study additive Gaussian noise models, for instance, the well known Rudin-Osher-Fatemi (ROF) model[1] and its extensions[2−4]. The problem can be stated as: Assume the true image u defined on $\Omega \subset \mathbb{R}^2$ is corrupted by some additive Gaussian noise n, the goal is to recover u from the observed data $f = u + n$. The ROF model is

$$\min\left\{E(u)=\int_\Omega |\nabla u|dx+\frac{\lambda}{2}\int_\Omega (u-f)^2dx\right\} \tag{1}$$

where the first term in the energy is the total variation regularization term, the second term is fidelity term and λ is a positive fidelity parameter. The ROF model performs well for cartoon like images since edge sharpness and location are well preserved. However, when denoising partly textured images, some textures as well as noise will be smoothed out. Many researchers have tried to overcome this drawback[5−9]. Gilboa-Zeevi-Sochen[6] proposed the local ROF model as

$$\min\left\{E(u)=\int_\Omega |\nabla u|dx+\frac{1}{2}\lambda(x)P_R(x)\right\} \tag{2}$$

where $P_R(x)$ is the local variance. This local model outperforms the original ROF model by keeping important textures.

In medical and astronomical imaging systems, which rely on photon detection as a basis of image formation, Poisson noise rather than Gaussian noise is frequently present in the images. Unlike Gaussian additive noise, Poisson noise is intensity dependent. In other words, bright pixels are statistically more corrupted by noise than dark pixels. The Poisson noise problem has been studied by wavelet method[10], Bayesian method[11], adaptive window approach[12] and variational methods. The variational methods for Poisson noise removal involve total variation regularization and a fidelity term related to Poisson distribution[13−15].

In a similar framework of the ROF model, Le-Chartrand-Asaki[14] (LCA) proposed a variational model for Poisson noise removal. Assume the observed image corrupted by Poisson noise is f, the aim is to recover the true image u. Assume the Poisson probability

$$P(u|f)=P_u(f)=\frac{e^{-u}u^f}{f!} \tag{3}$$

by Maximum a posteriori (MAP) method, they derived the energy functional

$$E(u)=\int_\Omega |\nabla u|dx+\lambda\int_\Omega (u-f\log u)dx \tag{4}$$

where λ is a positive parameter. This is called the constant LCA model later. The optimal parameter λ in Eq.(4) is difficult to choose. Like the ROF model, this model gives good denoising results for cartoon like images. However, it is not effective for denoising images containing textures since important textures as well as noise will be removed in the denoising process. In this paper, firstly we propose a scheme to choose the parameter λ automatically. Secondly, we propose a local fidelity term with spatial-varying parameters $\lambda(x)$, $x \in \Omega$. This local fidelity term automatically controls the extent of denoising according to image contents such that the textures can be well preserved.

Our paper is organized as follows. In Section II, we propose the varying scalar LCA model and the local LCA model. Then numerical implementation details are given in Section III. In Section IV, we display our experimental results and conduct

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some comparison. Finally, we conclude our paper in Section V.

II. The Proposed Model

Since the Poisson noise is dependent on the image intensity which is similar to the multiplicative noise case, we introduce the random variable $n = f/u$ to estimate the Poisson noise. Following the idea of Ref.[13], in which Gaussian distribution is assumed for general multiplicative noise with good results, we try to use this distribution on Poisson noise. As in Eq.(4), the intensity of the true image u is the expectation of the observed image f, so it is reasonable to assume that $n = f/u$ has mean 1. We suppose the standard deviation of n is σ, then

$$\int_\Omega \left(\frac{f}{u}-1\right)dx = 0 \quad \text{and} \quad \int_\Omega \left(\frac{f}{u}-1\right)^2 dx = |\Omega|\sigma^2 \tag{5}$$

where σ is a prior parameter. We estimate this parameter by trial and error through amounts of experiments.

1. Varying scalar LCA model

In the constant LCA model (5), λ is assumed to be a fixed constant. However, it is not easy to get a proper choice. To overcome this problem, we will derive an automatic procedure for λ. By standard variatinal method, the Euler-Lagrange equation of the constant LCA model (4) is given by

$$\text{div}\left(\frac{\nabla u}{|\nabla u|}\right) + \lambda \frac{f-u}{u} = 0 \tag{6}$$

where "div" denotes the divergence operator[14]. Then we multiply Eq.(6) by $(1 - f/u)$ and integrate on Ω which yield

$$\int_\Omega \text{div}\left(\frac{\nabla u}{|\nabla u|}\right)\left(1-\frac{f}{u}\right)dx - \lambda \int_\Omega \left(\frac{f}{u}-1\right)^2 dx = 0$$

Using the assumption Eq.(5), the scalar λ can be computed by

$$\lambda = \int_\Omega \text{div}\left(\frac{\nabla u}{|\nabla u|}\right)\left(1-\frac{f}{u}\right)\Big/(|\Omega|\sigma^2) \tag{7}$$

Therefore the solution of Eq.(6) can be obtained by iteratively evolving the negative gradient flow

$$\begin{cases} \dfrac{\partial u}{\partial t} = \text{div}\left(\dfrac{\nabla u}{|\nabla u|}\right) + \lambda \dfrac{f-u}{u}, \\ u|_{t=0} = f \end{cases}$$

with Neumann boundary condition $\dfrac{\partial u}{\partial \boldsymbol{N}} = 0$ on $\partial\Omega$ ($\boldsymbol{N}$ is the unit outward normal of the boundary $\partial\Omega$) and updating λ with Eq.(7) until convergence. This is called the varying scalar LCA model.

2. Local LCA model

We note in Eq.(4) that λ is a weighting parameter to balance the regularization term and the fidelity term. The regularization term asks that the image should be smooth and the fidelity term asks that the useful information in the noisy image f should be retained. Hence a small λ is suitable for denoising cartoon regions while a large λ is suitable for denoising texture regions. Since there are many features in an image such as cartoon, textures and small details. A global constant parameter λ may not be suitable for all the image features. Hence a local (spatial-varying) fidelity coefficient $\lambda(x)$ will be useful in order to restore different features in the denoising process. Here we adapt the idea in Ref.[6] which considers only additive Gaussian noise for multiplicative noise removal.

For each point $y \in \Omega$, the local fidelity is given by

$$E_y(u) = \lambda(y)\int_\Omega K(y-x)(u(x) - f(x)\log(u(x)))dx$$

where K is a 2-dimensional Gaussian kernel satisfying $\int_\Omega K(x)dx = 1$, $K(-x) = K(x)$. We propose to minimize

$$E(u,\lambda) = \int_\Omega |\nabla u|dx + \int_\Omega \lambda(y)E_y(u)dy \tag{8}$$

Using the symmetry of the Gaussian kernel K and the definition of convolution "$*$", we obtain

$$\begin{aligned} E(u,\lambda) &= \int_\Omega |\nabla u|dx + \int_\Omega\int_\Omega \lambda(y)K(y-x)(u(x) \\ &\quad - f(x)\log(u(x)))dxdy \\ &= \int_\Omega |\nabla u|dx + \int_\Omega (\int_\Omega \lambda(y)K(y-x)dy)(u(x) \\ &\quad - f(x)\log(u(x)))dx \\ &= \int_\Omega |\nabla u|dx + \int_\Omega (K*\lambda)(x)(u(x) - f(x)\log(u(x)))dx \end{aligned}$$

Using standard variational methods, we compute the Gateaux derivative of E about u in the direction v

$$\begin{aligned} \frac{\delta E}{\delta u}(v) &= \frac{d}{dt}|_{t=0}E(u+tv,\lambda) \\ &= \frac{d}{dt}|_{t=0}\int_\Omega |\nabla u + t\nabla v|dx \\ &\quad + \int_\Omega (K*\lambda)(u+tv - f\log(u+tv))dx \\ &= \int_\Omega \frac{\nabla u}{|\nabla u|}\nabla v dx + \int_\Omega (K*\lambda)\left(v - \frac{fv}{u}\right)dx \\ &= \int_\Omega \left[-\text{div}\left(\frac{\nabla u}{|\nabla u|}\right) + (K*\lambda)\left(1-\frac{f}{u}\right)\right] \\ &\quad \cdot v dx + \int_{\partial\Omega} \frac{\nabla u}{|\nabla u|}\cdot \boldsymbol{N} dx \end{aligned}$$

Then the Euler-Lagrange equation for u is

$$-\text{div}\left(\frac{\nabla u}{|\nabla u|}\right) + K*\lambda\left(1-\frac{f}{u}\right) = 0 \tag{9}$$

with Neumann boundary condition. Multiplying Eq.(9) by $(1 - f/u)$ and then integrating on Ω, and using the symmetry of K yields

$$\begin{aligned} \int_\Omega \text{div}\left(\frac{\nabla u}{|\nabla u|}\right)\left(1-\frac{f}{u}\right)dx &= \int_\Omega K*\lambda\left(\frac{f}{u}-1\right)^2 dx \\ &= \int_\Omega \lambda K*\left(\frac{f}{u}-1\right)^2 dx \end{aligned}$$

A sufficient condition for the above equalities is

$$\text{div}\left(\frac{\nabla u}{|\nabla u|}\right)\left(1-\frac{f}{u}\right) = \lambda K*\left(\frac{f}{u}-1\right)^2.$$

Then the function $\lambda(x)$ can be calculated by

$$\lambda(x) = Q(x)/S(x) \tag{10}$$

with

$$Q(x) = \text{div}\left(\frac{\nabla u(x)}{|\nabla u(x)|}\right)\left(1 - \frac{f(x)}{u(x)}\right), \quad S(x) = K * \left(\frac{f}{u} - 1\right)^2 (x)$$

We note that $S(x)$ represents the local variance of the noise $n = f/u$.

Following Ref.[6], we regard $S(x)$ as a prior in the updating formula Eq.(10). Giving a proper prior allows us to enhance the denoising performance. We estimate $S(x)$ as follows: First we preprocess the noisy image by the varying scalar LCA model (or the constant LCA model) and assume the solution be $\tilde{u}$, which is a good approximation of the true image. Calculating the noise local variance related to $\tilde{u}$ as

$$LV(x) = K * (r - \bar{r})^2$$

where $r = f/\tilde{u} - 1$, $\bar{r}$ denotes the mean of r, $f/\tilde{u}$ is regarded as the noise residue. Then we assign

$$S(x) \approx \frac{\sigma^4}{LV(x)} \tag{11}$$

The reason is: If the image is cartoon-like, then the varying scalar model gives high quality denoising result $\tilde{u}$ and the noise residue $n \approx f/\tilde{u}$. The local variance is almost a global constant, $LV(x) = \sigma^2$, and then $S(x) = \sigma^2$ globally. Hence the local LCA model reduces to the constant LCA model which is good for cartoon denoising. In the case of images with textures (most natural images belong to this type), textures as well as noise will be filtered and included in the noise residue $f/\tilde{u}$. Then the local variance $LV(x)$ is approximately the sum of local variances of the textures ($LVtex$) and the noise (σ^2) in the noise residue. Hence the textured regions are characterized by high local variance of the noise residue. In order to preserve such textures, the fidelity should be increased over these regions. By Eq.(11)

$$S(x) = \frac{\sigma^4}{LVtex + \sigma^2} < \sigma^2$$

Then the local fidelity coefficient in texture regions is larger than in the cartoon regions and thus the texture regions are better preserved.

Therefore we conclude that the solution of energy Eq.(8) can be found by iteratively evolving the negative gradient flow

$$\begin{cases} \dfrac{\partial u}{\partial t} = \text{div}\left(\dfrac{\nabla u}{|\nabla u|}\right) + K * \lambda \dfrac{f - u}{u}, \\ u|_{t=0} = f \end{cases} \tag{12}$$

with Neumann boundary condition, while updating $\lambda(x)$ with Eq.(10) and updating $S(x)$ with Eq.(12) until convergence. It is called the local LCA model in this paper.

The above idea of extending the constant fidelity LCA model to the varying scalar LCA model and then to the local LCA model is easy to generalize to other variational denoising models such as the ROF model and its extensions. When it is applied to the ROF model, we have a more natural explanation to get the model in Ref.[6].

III. Numerical Implementation

For numerical implementation, we use finite differences schemes. To avoid division by zero, the divergence term $\text{div}\left(\frac{\nabla u}{|\nabla u|}\right)$ in Eq.(12) is replaced by $\text{div}\left(\frac{\nabla u}{\sqrt{|\nabla u|^2 + \varepsilon^2}}\right)$ where $\varepsilon > 0$. This term is calculated by

$$\frac{u_{x^1x^1}(u_{x^2}^2 + \varepsilon^2) - 2u_{x^1}u_{x^2}u_{x^1x^2} + u_{x^2x^2}(u_{x^1}^2 + \varepsilon^2)}{(u_{x^1}^2 + u_{x^2}^2 + \varepsilon^2)^{3/2}} \tag{13}$$

where $x = (x^1, x^2) \in \Omega$. In Eq.(13), the derivatives are all approximated by central difference. Explicit schemes are used in all the cases. The Gaussian kernel K is given by a Gaussian window of standard deviation $\sigma_w = 5$ with window size $4\sigma_w + 1$. We set $\varepsilon = 1$, time step $\tau = 0.1$ and $\sigma = 0.1$ for all test images.

IV. Experimental Results

We test our algorithms with a synthetic image and two standard test images which all contain textures. The Poisson noise is simulated by Matlab image processing toolbox. For the constant LCA model, we choose the optimal parameter λ by trial and error. The experiments show that the local LCA model preserves textures better than the other models. The constant LCA model and the varying scalar LCA model have similar performance.

In order to quantify the denoising performance, we compare the Mean-absolute-error (MAE) and Signal-to-noise ratio (SNR) of the denoised images for different models. Table 1 shows the MAE and SNR for all the test images. “Noisy”, “Const”, “Scalar”, “Local” and “AA” in Table 1 represent the constant LCA model, the varying scalar LCA model and the local LCA model respectively. From Table 1, we conclude that the local LCA model is the best since it has the lowest MAE and highest SNR, while the other two models have similar performance.

Table 1. Performance comparison of the three models

	MAE				SNR			
Image	Noisy	Const	Scalar	Local	Noisy	Const	Scalar	Local
Mosaic	9.1	7.0	7.1	5.6	13.0	13.2	13.3	14.5
Lena	8.7	4.1	4.2	3.9	12.6	17.8	17.6	18.7
Barbara	8.3	5.9	5.8	5.3	13.5	15.2	15.4	16.6

Then we test three images contain textures showed in Fig.1(a), Fig.2(a) and Fig.3(a). In Fig.1, we test the synthetic mosaic image with Poisson noise in Fig.1(b). Fig.1(c) shows that the local fidelity function $\lambda(x)$ can almost distinguish the cartoon regions from texture regions. The denoising results of the three models are displayed in Fig.1($d-f$). The corresponding noise residues of the models are displayed in Fig.1($g-i$). Visually, the results of the local LCA model are the best among all. In the upper left and bottom right regions with constant values, the result of the local LCA model is good. However, there are some blocks in the results of the other two models. In the regions of textures, the local LCA model preserves more textures. The noise residue is calculated by the formula f/u and then multiplied by 128 for display throughout the paper. The criterion is: the noise residue contains less useful information such as textures and edges, the denoising performance is better. It is clear that the noise residue of the local LCA model contains least textures than others.

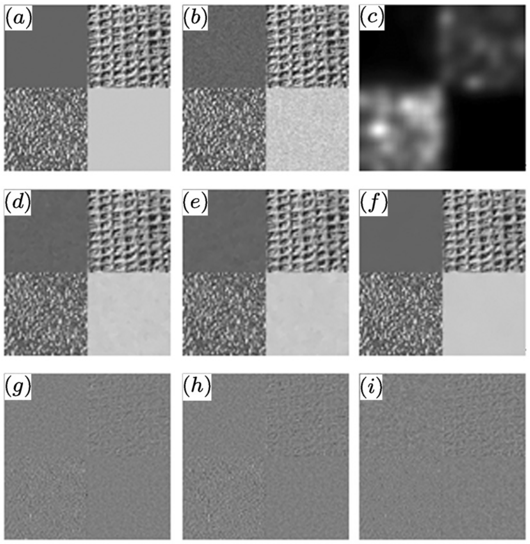

Fig. 1. (a) The synthetic mosaic image; (b) Image corrupted by Poisson noise; (c) Local fidelity coefficient $\lambda(x)$; (d) Result of constant LCA model with $\lambda = 16$; (e) Result of the varying scalar LCA model; (f) Result of the local LCA model; $(g-i)$ The noise residue of $(d-f)$

Lena image with Poisson noise is tested in Fig.2. Let us take a careful look at the region of hair which is detected as texture in the $\lambda(x)$ image in Fig.2(c). From Fig.3$(d-f)$, we see that the local LCA model keeps these textures best among the three models, while the constant LCA model and the varying scalar LCA model are similar. It is obvious that the noise residue of the local LCA model Fig.3(i) contains less hair textures than those of the other models.

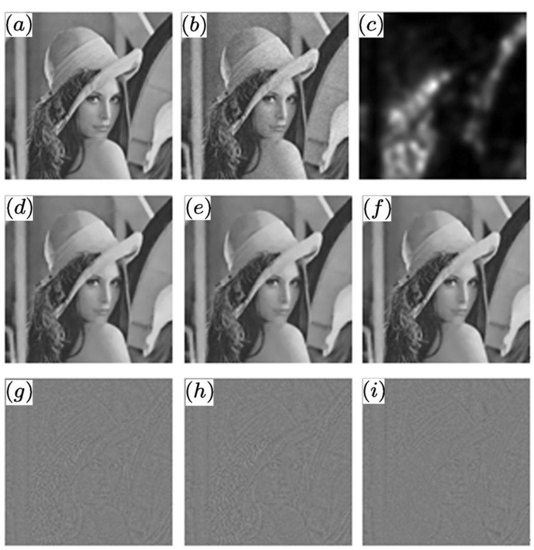

Fig. 2. (a) Lena image; (b) Image corrupted by Poisson noise; (c) Local fidelity coefficient $\lambda(x)$; (d) Result of constant LCA model with $\lambda = 7$; (e) Result of the varying scalar LCA model; (f) Result of the local LCA model; $(g-i)$ The noise residue of $(d-f)$

Finally we test Barbara image with Poisson noise in Fig.3. With a careful comparison of the scarf, we find that the local LCA model gives the best result among the three models. The noise residue of the local LCA model Fig.3(i) contains the least textures of the scarf among Fig.3$(g-i)$.

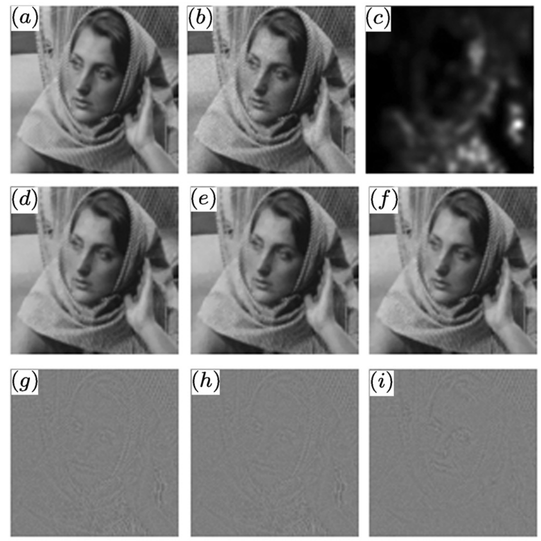

Fig. 3. (a) Barbara image; (b) Corrupted by Poisson noise; (c) Local fidelity coefficient $\lambda(x)$; (d) Result of constant LCA model with $\lambda = 10$; (e) Result of the varying scalar LCA model; (f) Result of the local LCA model; $(g-i)$ The noise residue of $(d-f)$

V. Conclusion

To enhance the quality of LCA model when processing partly textured images polluted by Poisson noise, we propose the local LCA model by introducing a local fidelity term with spatial-varying parameters. The local LCA model can adjust the fidelity according to the image information, texture regions with more fidelity and cartoon regions with less fidelity. Further study will be focused on generalizing the method to other existing variational image processing methods in denoising, deblurring and segmentation.

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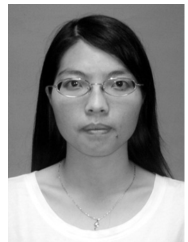

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