On rational proper mappings among generalized complex balls

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Dedicated to Professor Ngaiming Mok on his 60th birthday

Abstract

We introduce the notion of *multiplier*, a real-valued bihomogeneous polynomial $M_F \in \mathbb{C}[z_1, \bar{z}_1, \dots, z_{r+s}, \bar{z}_{r+s}]$ canonically associated to a rational proper map F from a generalized ball $D_{r,s}$ to another generalized ball. We prove that the multiplier M_F essentially determines the map F and hence one can study the structure of rational proper mappings among generalized balls through the multiplier. We use the multiplier to study degree-2 rational proper maps from $D_{2,2}$ to an arbitrary $D_{r,s}$, demonstrating first of all that one may confine itself to the cases where $r, s \geq 2$ and $r + s \leq 10$ without loss of generality. Then, we show that for each maximal case, i.e. whenever $r + s = 10$, there exists a real-parameter family of non-equivalent degree-2 holomorphic proper maps. Finally, we give a complete description of all degree-2 rational proper maps from $D_{2,2}$ to $D_{3,3}$, which is the minimal case where there are non-standard mappings.

1 Introduction

A *generalized complex ball*, or simply *generalized ball*, denoted by $D_{r,s}$, is a domain on the complex projective space generalizing the complex unit ball. Explicitly, for any pair of positive integers r, s, it is defined as

$$D_{r,s} = \left\{ [z_1, \dots, z_{r+s}] \in \mathbb{P}^{r+s-1} : \sum_{j=1}^{r} |z_j|^2 > \sum_{j=r+1}^{r+s} |z_j|^2 \right\}.$$

When $r = 1$, we see that it is just the ordinary unit ball $\mathbb{B}^s$ embedded in $\mathbb{P}^s$. In general, they are also examples of the so-called *flag domains* on the complex projective

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space, when the latter is regarded as a flag manifold. In other words, it is an open orbit on $\mathbb{P}^n$ of some real form of $SL(n;\mathbb{C})$ (in this case, it is $SU(r,s)$, where $r+s=n$).

As a generalization of the unit ball, rigidity problems for holomorphic mappings on $D_{r,s}$ have been studied by a number of people, especially those in Cauchy-Riemann Geometry. In particular, Baouendi-Huang [BH] and Baouendi-Ebenfelt-Huang [BEH] have studied the rigidity for local proper holomorphic mappings among generalized balls. By these works, it is now known that unlike the case for the ordinary unit balls, the co-dimension is not the key issue for the problem. For example, Baouendi-Huang [BH] proved that if $r, s, s' \geq 2$ and $s' \geq s$, any local proper holomorphic map from $D_{r,s}$ to $D_{r,s'}$ is the restriction of a standard embedding and hence linear. (For a different proof using cycle spaces, see [Ng2].) On the other hand, if we allow r', s' to be sufficiently greater than r, s, it is easy to construct non-linear examples of proper holomorphic maps from $D_{r,s}$ to $D_{r',s'}$. Under certain restrictions on (r,s) and (r',s'), Baouendi-Ebenfelt-Huang [BEH] have proven partial rigidity for local proper holomorphic maps using methods in Cauchy-Riemann Geometry and the second author of the current article on the other hand has proven full rigidity for the same pairs of generalized balls in the global setting by using cycle spaces [Ng1].

As mentioned, for a given $D_{r,s}$, there are non-linear proper holomorphic maps from $D_{r,s}$ to $D_{r',s'}$ if r', s' are large enough. Thus, the classification problem appears naturally. The case for the ordinary unit balls has been studied quite a lot, see for example, D'Angelo [Da1, Da2], Faran [Fa], Faran-Huang-Ji-Zhang [FHJZ], Huang-Ji [HJ], Huang-Ji-Xu [HJX]. The general case, which is a natural problem by itself, is motivated by its link to the proper holomorphic mappings among Type-I irreducible bounded symmetric domains. For the detail of this linkage, we refer the readers to [Ng4]. Roughly speaking, a proper holomorphic map from $D_{r,s}$ to $D_{r',s'}$ will induce a proper holomorphic map from $\Omega_{r,s}$ to $\Omega_{r',s'}$, where $\Omega_{r,s}$ and $\Omega_{r',s'}$ are Type-I irreducible bounded symmetric domains. (See [Se] for some explicit examples.) Conversely, it has also been shown that the known examples of proper holomorphic maps among Type-I irreducible bounded symmetric domains also induce proper maps (which may be only meromorphic) among generalized balls. Moreover, it is known that any global holomorphic map among generalized balls must be rational [Ng1] and the same proof actually also works for meromorphic maps. Thus, any classification results for rational proper maps among $D_{r,s}$ will give information about the structure of proper holomorphic maps among $\Omega_{r,s}$. In the past the study of the latter mappings has remained rather unexplored as the singularities on the boundaries of $\Omega_{r,s}$ of higher rank hinder the application of the usual analytic methods, like the Chern-Moser normal form, etc.

With the above motivation, we study in the current article the rational proper maps among generalized balls (which will be defined more precisely later). We introduce the notion of *multiplier*, a real-valued bihomogeneous polynomial $M_F \in \mathbb{C}[z_1, \bar{z}_1, \ldots, z_{r+s}, \bar{z}_{r+s}]$ associated to any given rational proper map F from $D_{r,s}$ to another generalized ball. We will show that the multiplier M_F essentially determines the rational proper map F (Theorem 3.6) and thus one may study the structure of rational proper maps among generalized balls through the multiplier. Along this line, it is natural for us to study rational proper maps with a fixed degree from a fixed $D_{r,s}$ while allowing the target generalized ball to be variable. As a first exploration, we confine ourselves to degree-2 rational proper maps defined on $D_{2,2}$, which is the simplest generalized ball other than the ordinary unit balls. It turns out that in this case there are already plenty of non-trivial examples of rational (or holomorphic) proper maps. Using the multiplier, we first demonstrate that there is a real-parameter family of non-equivalent holomorphic proper maps from $D_{2,2}$ to $D_{r,s}$ whenever $r, s \geq 2$ and $r + s = 10$ (Theorem 4.2). Here we remark that for studying degree-2 rational proper maps from $D_{2,2}$ to an arbitrary $D_{r,s}$, it suffices to consider only the cases where $r + s \leq 10$ (Proposition 4.1). Finally, we determine all the multipliers that can give rise to degree-2 rational proper maps from $D_{2,2}$ to $D_{3,3}$ and hence obtain a complete description of all degree-2 rational proper maps from $D_{2,2}$ to $D_{3,3}$ (Theorem 4.3). We remark that this is the minimal case where there are non-linear rational proper maps.

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2 Notations and definitions

Let $H_{r,s}$ be the diagonal matrix such that first r diagonal entries are $+1$ and the remaining entries are -1. Define the indefinite inner product of signature (r, s) on $\mathbb{C}^{r+s}$ as

$$\langle z, w\rangle_{r,s} = \bar{w}H_{r,s}z^t = z_1\bar{w}_1 + \cdots + z_r\bar{w}_r - z_{r+1}\bar{w}_{r+1} - \cdots - z_{r+s}\bar{w}_{r+s},$$

where $z = (z_1, \ldots, z_{r+s})$ and $w = (w_1, \ldots, w_{r+s})$. We also write the indefinite norm as $\|z\|^2_{r,s} = \langle z, z\rangle_{r,s}$.

Denote by $M(m, n; \mathbb{C})$ the set of m-by-n complex matrices. The transpose of a matrix A will be denoted by A^t. Let $r, s \geq 0$, we denote by $U(r, s) \subset M(r+s, r+s; \mathbb{C})$ the generalized unitary group of signature (r, s), i.e. $U \in U(r, s)$ if and only if $\bar{U}^t H_{r,s}U = H_{r,s}$. We also write $U(r, 0)$ as $U(r)$.

Let $F : \mathbb{P}^{r+s-1} \dashrightarrow \mathbb{P}^{r'+s'-1}$ be a rational map. Write $F = [f_1, \ldots, f_{r'+s'}]$, where f_j, $1 \leq j \leq r' + s'$, are relatively prime homogeneous polynomials in the homogeneous coordinates $[z_1, \ldots, z_{r+s}]$ of $\mathbb{P}^{r+s-1}$. The polynomials f_j are only determined by F up to a common scalar multiple. With a slight abuse of notation, we will write

$$\|F\|^2_{r',s'} = \sum_{j=1}^{r'} |f_j|^2 - \sum_{j=r'+1}^{r'+s'} |f_j|^2$$

for a given F, although $\|F\|^2_{r',s'}$ is only well defined up to a scalar factor (depending on our choice for f_j representing F).

Definition 2.1. *Let $r, s \geq 1$ be positive integers. The domain $D_{r,s} \subset \mathbb{P}^{r+s-1}$, defined by*

$$D_{r,s} = \{z = [z_1, \ldots, z_{r+s}] \in \mathbb{P}^{r+s-1} : \|z\|^2_{r,s} > 0\},$$

is called a ***generalized ball****.*

Definition 2.2. *A rational map $F : \mathbb{P}^{r+s-1} \dashrightarrow \mathbb{P}^{r'+s'-1}$ is called a* ***rational proper map*** *from $D_{r,s}$ to $D_{r',s'}$ if there is a connected open set $\mathcal{U} \subset \mathbb{P}^{r+s-1}$ in the complex topology, with $\mathcal{U} \cap \partial D_{r,s} \neq \emptyset$, such that $F : \mathcal{U} \to \mathbb{P}^{r'+s'-1}$ is holomorphic, $F(\mathcal{U} \cap D_{r,s}) \subset D_{r',s'}$ and $F(\mathcal{U} \cap \partial D_{r,s}) \subset \partial D_{r',s'}$. We remark that according to our definition, the image of $D_{r,s}$ under F may not lie entirely inside $D_{r',s'}$.*

In this article, we will denote a rational proper map F from $D_{r,s}$ to $D_{r',s'}$ by

$$F : D_{r,s} \dashrightarrow \mathbb{P}^{r'+s'-1} \supset D_{r',s'}.$$

3 Multiplier associated to a rational proper map

Throughout this section, we let $F : D_{r,s} \dashrightarrow \mathbb{P}^{r'+s'-1} \supset D_{r',s'}$ be a rational proper map and write $F = [f_1, \ldots, f_{r'+s'}]$, where f_j, $1 \leq j \leq r' + s'$, are relatively prime homogeneous polynomials of the same degree in $\mathbb{C}[z_1, \ldots, z_{r+s}]$.

3.1 Multiplier

Proposition 3.1. *There exists some real-valued bihomogeneous polynomial $M_F \in \mathbb{C}[z_1, \bar{z}_1, \ldots, z_{r+s}, \bar{z}_{r+s}]$ such that*

$$\|F\|^2_{r',s'} = \|z\|^2_{r,s} M_F.$$

Furthermore, M_F is uniquely determined up to a positive scalar factor.

Proof. At any point $p \neq \mathbf{0} \in \mathbb{C}^{r+s}$, we have the differential $d\|z\|^2_{r,s} \neq 0$. Let $[p] \in \mathbb{P}^{r+s-1}$ be the projectivization of $p \in \mathbb{C}^{r+s}$. Since F is a rational proper map, there is some neighborhood $\mathcal{U}$ containing $[p] \in \mathbb{P}^{r+s-1}$ such that $\|F(z)\|^2_{r',s'} = 0$ whenever $\|z\|^2_{r,s} = 0$. Hence, by shrinking $\mathcal{U}$ if necessary, there exists a real analytic function

$\rho(z,\bar{z})$ on $\mathcal{U}$ such that $\|F\|^2_{r',s'} = \|z\|^2_{r,s}\,\rho$ on $\mathcal{U}$. Polarizing, we get $\langle F(z), F(w)\rangle_{r',s'} = \langle z, w\rangle_{r,s}\,\rho(z,w)$. Using the identity theorem of holomorphic functions, we then see that $\langle F(z), F(w)\rangle \in \mathbb{C}[z_1, w_1, \ldots, z_{r+s}, w_{r+s}]$ vanishes on the zero set of the irreducible polynomial $\langle z, w\rangle_{r,s}$. Therefore, $\langle z, w\rangle_{r,s}$ is an irreducible factor of $\langle F(z), F(w)\rangle$. The proposition now follows when we set back $w = \bar{z}$.

It is clear that M_F is real valued and bihomogeneous since it is the quotient of two norm-squares and thus its terms contain the same number of holomorphic variables and conjugate-holomorphic variables. Finally, M_F is uniquely determined by F up to a positive scalar factor since the same is true for $\|F\|^2_{r,s}$. □

For a given F, although M_F, like $\|F\|^2_{r',s'}$, is only well defined up to a scalar factor, this constant factor will be immaterial in the forthcoming discussion. We will thus, with a slight abuse of language, call the polynomial M_F the *multiplier* of F. We have the following representation for M_F.

Proposition 3.2. *Let deg$(F) = k$ and $N = \binom{r+s+k-2}{k-1}$. Let $\mathbf{Z}_{k-1}$ be the column vector consisting of all the monomials of degree $(k-1)$ in $z_1, \ldots, z_{r+s}$ arranged in the lexicographical order. Then there exists an $N \times N$ Hermitian matrix A_F such that $M_F(z,\bar{z}) = \bar{\mathbf{Z}}^t_{k-1} A_F \mathbf{Z}_{k-1}$..*

Proof. As a polynomial in $\mathbb{C}[z_1, \bar{z}_1, \ldots, z_{r+s}, \bar{z}_{r+s}]$, we have $\deg(M_F) = 2k-2$. Furthermore, as just mentioned above, every term of M_F contains the same number of holomorphic variables and conjugate-holomorphic variables. Using the standard multi-index notation, we can therefore write

$$M_F(z,\bar{z}) = \sum_{|I|=|J|=k-1} a_{I\bar{J}} z^I \bar{z}^J = \bar{\mathbf{Z}}^t_{k-1} A_F \mathbf{Z}_{k-1}$$

if we define the (J, I)-th element of A_F to be $a_{I\bar{J}}$. Finally, as M_F is real-valued, we have $a_{J\bar{I}} = \overline{a_{I\bar{J}}}$ and hence A_F is Hermitian. □

3.2 Unique determination of proper maps by the multiplier

In what follows, we will denote by $H_{m,n,p}$ the diagonal matrix whose the first m diagonal entries are $+1$, , and the next n diagonal entries are -1, and the last p diagonal entries are 0. Thus, $H_{m,n,0} = H_{m,n}$ according to our previous convention.

Lemma 3.3. *Suppose*

$$\sum_{j=1}^{m} |a_j|^2 - \sum_{j=m+1}^{m+n} |a_j|^2 = \sum_{j=1}^{m'} |b_j|^2 - \sum_{j=m'+1}^{m'+n'} |b_j|^2, \tag{1}$$

where a_j, b_j are degree-k homogeneous polynomials in $\mathbb{C}[z_1, \ldots, z_{r+s}]$ and $\{a_1, \ldots, a_{m+n}\}$ are linearly independent. Then, $m \le m'$, $n \le n'$, and there exist a unique non-negative *integer $q \le m'+n'-m-n$, and degree-k homogeneous polynomials $c_1,\ldots,c_q \in \mathbb{C}[z_1,\ldots,z_{r+s}]$, and $W \in M(m'+n', m+n+q;\mathbb{C})$ satisfying $\bar{W}^t H_{m',n'} W = H_{m,n,q}$, such that*

$$(b_1,\ldots,b_{m'+n'})^t = W(a_1,\ldots,a_{m+n},c_1,\ldots,c_q)^t.$$

Moreover, if $\{b_1,\ldots,b_{m'+n'}\}$ are also linear independent, then $m=m'$, $n=n'$ and there exists $V \in U(m,n)$ such that

$$(b_1,\ldots,b_{m+n})^t = V(a_1,\ldots,a_{m+n})^t.$$

Proof. Rewrite the equation as

$$\sum_{j=1}^{m}|a_j|^2 + \sum_{j=m'+1}^{m'+n'}|b_j|^2 = \sum_{j=1}^{m'}|b_j|^2 + \sum_{j=m+1}^{m+n}|a_j|^2.$$

It is a standard fact (see [Ng3] for a proof) that there exists a matrix M such that

$$(a_1,\ldots,a_m,b_{m'+1},\ldots,b_{m'+n'})^t = M(b_1,\ldots,b_{m'},a_{m+1},\ldots,a_{m+n})^t.$$

In particular, for every $j\in\{1,\ldots,m\}$, there are complex numbers $\{\mu_{j1},\ldots,\mu_{jm'}\}$ and $\{\nu_{j1},\ldots,\nu_{jn}\}$ such that

$$a_j = \mu_{j1}b_1+\cdots+\mu_{jm'}b_{m'}+\nu_{j1}a_{m+1}+\cdots+\nu_{jn}a_{m+n}.$$

Equivalently, for every $j\in\{1,\ldots,m\}$,

$$a_j-\nu_{j1}a_{m+1}-\cdots-\nu_{jn}a_{m+n} = \mu_{j1}b_1+\cdots+\mu_{jm'}b_{m'}.$$

Now if $m>m'$, then we can find a non-zero vector $(\xi_1,\ldots,\xi_m)\in\mathbb{C}^m$ such that

$$\xi_1\mu_{1\ell}+\cdots+\xi_m\mu_{m\ell}=0$$

for every $\ell\in\{1,\ldots,m'\}$. This implies

$$\sum_{j=1}^{m}\xi_j(a_j-\nu_{j1}a_{m+1}-\cdots-\nu_{jn}a_{m+n}) = \sum_{j=1}^{m}\xi_j(\mu_{j1}b_1+\cdots+\mu_{jm'}b_{m'})=0.$$

As the functions a_i are linearly independent, we have $\xi_j=0$, which is a contradiction. So $m\le m'$. Multiplying Eq (1) by -1, we see immediately that we also have $n\le n'$.

We now first settle the case where $\{b_1,\ldots,b_{m'+n'}\}$ are also linearly independent. By symmetry, we must also have $m'\le m$ and $n'\le n$, and hence $m=m'$, $n=n'$. Moreover, in the argument above, the matrix $(\mu_{jk})_{1\le j,\,k\le m}$ is invertible and we deduce that every b_j, where $1 \leq j \leq m$, can be written as a linear combination in a_k. By symmetry, the same holds true for $m+1 \leq j \leq m+n$. Therefore, we can write

$$(b_1, \ldots, b_{m+n})^t = V(a_1, \ldots, a_{m+n})^t,$$

for some invertible matrix V of rank $m+n$. Now by Eq (1) and the fact that $\{a_1, \ldots, a_{m+n}\}$ are linearly independent, we see that $\bar{V}^t H_{m,n} V = H_{m,n}$ and thus V preserves the indefinite inner product of signature (m,n) on $\mathbb{C}^{m+n}$. Hence, $V \in U(m,n)$.

Now suppose that $\{b_1, \ldots, b_{m'+n'}\}$ are linearly dependent. Choose p linearly independent degree-k homogeneous polynomials $d_1, \ldots, d_p \in \mathbb{C}[z_1, \ldots, z_{r+s}]$ such that

$$(b_1, \ldots, b_{m'+n'})^t = X(d_1, \ldots, d_p)^t$$

for some matrix $X \in M(m'+n', p; \mathbb{C})$. Consider the Hermitian matrix $\bar{X}^t H_{m',n'} X$. There exists a unitary matrix $U \in U(p)$ such that

$$\bar{U}^t \bar{X}^t H_{m',n'} X U = H_{\tilde{m},\tilde{n},q},$$

where $\tilde{m} + \tilde{n} + q = p$. Now if we define

$$(e_1, \ldots, e_p)^t := \bar{U}^t (d_1, \ldots, d_p)^t,$$

then we have

$$\begin{aligned}
&(\bar{b}_1, \ldots, \bar{b}_{m'+n'}) H_{m',n'} (b_1, \ldots, b_{m'+n'})^t \\
=&(\bar{d}_1, \ldots, \bar{d}_p) \bar{X}^t H_{m',n'} X (d_1, \ldots, d_p)^t \\
=&(\bar{d}_1, \ldots, \bar{d}_p) U \bar{U}^t \bar{X}^t H_{m',n'} X U \bar{U}^t (d_1, \ldots, d_p)^t \\
=&(\bar{e}_1, \ldots, \bar{e}_p) H_{\tilde{m},\tilde{n},q} (e_1, \ldots, e_p)^t
\end{aligned}$$

That is,

$$\sum_{j=1}^{m'} |b_j|^2 - \sum_{j=m'+1}^{m'+n'} |b_j|^2 = \sum_{j=1}^{\tilde{m}} |e_j|^2 - \sum_{j=\tilde{m}+1}^{\tilde{m}+\tilde{n}} |e_j|^2.$$

Combining with Eq (1), and the fact that $\{a_1, \ldots, a_{m+n}\}$ and $\{e_1, \ldots, e_{\tilde{m}+\tilde{n}}\}$ are linearly independent, it follows from our previous conclusion that $m = \tilde{m}$, $n = \tilde{n}$, and there exists $Y \in U(m,n)$ such that

$$(e_1, \ldots, e_{m+n})^t = Y(a_1, \ldots, a_{m+n})^t.$$

Finally, if we let $Z = \begin{bmatrix} Y & 0 \\ 0 & I_p \end{bmatrix}$, where I_p is the identity matrix of rank p, and let $W = XUZ$, then we have

$$\bar{W}^t H_{m',n'} W = H_{m,n,q}$$

and

$$(b_1, \ldots, b_{m'+n'})^t = W(a_1, \ldots, a_{m+n}, c_1, \ldots, c_q)^t,$$

where $c_j = e_{m+n+j}$, $j \in \{1, \ldots, q\}$. Since $m + n + q = p \leq m' + n'$, we get

$$q \leq m' + n' - m - n.$$

□

Given a rational proper holomorphic map F between two generalized balls, its multiplier is uniquely determined by F (up to a scalar factor). On the other hand, we are now going to see how an arbitrary real-valued bihomogeneous polynomial in $\mathbb{C}[z_1, \bar{z}_1, \ldots, z_{r+s}, \bar{z}_{r+s}]$ gives rise to a rational proper map from $D_{r,s}$. To start with, we will need the notion of signature of real analytic functions on $\mathbb{C}^n$. For our purpose, we only need to restrict ourselves to the real-valued bihomogeneous polynomials in $\mathbb{C}[z_1, \bar{z}_1, \ldots, z_{r+s}, \bar{z}_{r+s}]$.

Let $k \in \mathbb{N}$ be arbitrary and h be a real-valued bihomogeneous polynomial in $\mathbb{C}[z_1, \bar{z}_1, \ldots, z_{r+s}, \bar{z}_{r+s}]$ of bidegree (k, k). Using the standard multi-index notation, we write $h(z, \bar{z}) = \sum_{|I|=|J|=k} a_{I\bar{J}} z^I \bar{z}^J$, where $z = (z_1, \ldots, z_{r+s})$. Since h is real, the coefficients $a_{I\bar{J}}$ constitute an $N \times N$ Hermitian matrix, where $N = \binom{r+s+k-1}{k}$. Diagonalize the matrix $(a_{I\bar{J}})$ as

$$a_{I\bar{J}} = \sum_{|K|=|L|=k} u_{IK} d_{K\bar{L}} \overline{u_{JL}},$$

where (u_{IK}) is a unitary matrix and $(d_{K\bar{L}})$ is a diagonal matrix whose diagonal entries are eigenvalues the of $(a_{I\bar{J}})$. Then by considering the polynomials $\sum_{|I|=k} u_{IK} z^I$, one sees that

$$h(z, \bar{z}) = \sum_{j=1}^{R} |h_j^+|^2 - \sum_{j=1}^{S} |h_j^-|^2 \tag{2}$$

for some homogeneous polynomials h_j^+, h_j^- in $(z_1, \ldots, z_{r+s})$ of degree k, where R and S are the number of positive and negative eigenvalues of $(a_{I\bar{J}})$ respectively. We will call the pair (R, S) the *signature* of h. By construction, the polynomials h_j^+, h_j^- are linearly independent.

Now fix a generalized ball $D_{r,s}$. We make the following definition for multipliers on $D_{r,s}$.

Definition 3.4. *Let $M(z, \bar{z}) \in \mathbb{C}[z_1, \bar{z}_1, \ldots, z_{r+s}, \bar{z}_{r+s}]$ be a real-valued bihomogeneous polynomial, where $z = (z_1, \ldots, z_{r+s})$. If there is a connected open set $\mathcal{U} \subset \mathbb{P}^{r+s-1}$ in the complex topology, such that $\mathcal{U} \cap \partial D_{r,s} \neq \emptyset$ and $M > 0$ on $\mathcal{U} \cap D_{r,s}$, then we call $M(z, \bar{z})$ a* ***multiplier*** *on $D_{r,s}$.*

Let $M \in \mathbb{C}[z_1, \bar{z}_1, \ldots, z_{r+s}, \bar{z}_{r+s}]$ be an arbitrary multiplier on $D_{r,s}$. Consider the real-valued bihomogeneous polynomial $h_M(z, \bar{z}) := \|z\|^2_{r,s} M(z, \bar{z})$, where $z = (z_1, \ldots, z_{r+s})$. If M is of bidegree (k, k), then h_M is of bidegree $(k+1, k+1)$. Thus, h_M has a decomposition (as that in Eq. 2):

$$h_M(z, \bar{z}) = \sum_{j=1}^{R} |h^+_{M,j}|^2 - \sum_{j=1}^{S} |h^-_{M,j}|^2.$$

In particular, if we define the map $F_M : D_{r,s} \to D_{R,S}$, where

$$F_M = [h^+_{M,1}, \ldots, h^+_{M,R}, h^-_{M,1}, \ldots, h^-_{M,S}], \tag{3}$$

then F_M is a rational proper map from $D_{r,s}$ to $D_{R,S}$. Furthermore, the image of F_M is not contained in any proper linear subspace.

Definition 3.5. *The map F_M is called a* ***canonical rational proper map*** *associated to M.*

By our construction, for a given multiplier M on $D_{r,s}$, the target generalized ball of any canonical rational map associated to M is uniquely determined and the canonical rational maps are unique up to the automorphisms of the target. We are going to show that any rational proper map from $D_{r,s}$ to an arbitrary generalized ball whose multiplier is equal to M can be factored through F_M or a trivial modification of F_M (which will be made precise below). For such purpose, we extend the definition of generalized balls and let

$$D_{r,s,q} = \left\{ [z_1, \ldots, z_{r+s+q}] \in \mathbb{P}^{r+s+q-1} : \sum_{j=1}^{r} |z_j|^2 > \sum_{j=r+1}^{r+s} |z_j|^2 \right\}.$$

In addition, we extend the notion of rational proper maps to $D_{r,s,q}$ naturally.

Theorem 3.6. *Let $M \in \mathbb{C}[z_1, \bar{z}_1, \ldots, z_{r+s}, \bar{z}_{r+s}]$ be an arbitrary multiplier on $D_{r,s}$ of bi-degree (k, k) and let the signature of $h_M := \|z\|^2_{r,s} M$ be (R, S), where $z = (z_1, \ldots, z_{r+s})$. Let $F_M : D_{r,s} \dashrightarrow \mathbb{P}^{R+S-1} \supset D_{R,S}$ be a canonical rational proper map associated to M and write $F_M = [f^M_1, \ldots, f^M_{R+S}]$. If $F : D_{r,s} \dashrightarrow \mathbb{P}^{r'+s'-1} \supset D_{r',s'}$ is a rational proper map such that its multiplier is equal to M, then $r' \geq R$, $s' \geq S$ and there exist a unique non-negative integer $Q \leq r' + s' - R - S$, and degree-k homogeneous polynomials $\{\psi_1, \ldots, \psi_Q\} \subset \mathbb{C}[z_1, \ldots, z_{r+s}]$, such that there is a factorization $F = H \circ \widetilde{F}$, where $H : D_{R,S,Q} \to D_{r',s'}$ is a linear proper embedding, and $\widetilde{F} : D_{r,s} \dashrightarrow \mathbb{P}^{R+S+Q-1} \supset D_{R,S,Q}$ is a rational proper map, with $\widetilde{F} = [f^M_1, \ldots, f^M_{R+S}, \psi_1, \ldots, \psi_Q]$.*

Proof. Following the previous notations as in Eq (3), we let

$$F_M = [h^+_{M,1}, \ldots, h^+_{M,R}, h^-_{M,1}, \ldots, h^-_{M,S}],$$

where

$$h_M(z,\bar{z}) = \sum_{j=1}^{R} |h^+_{M,j}|^2 - \sum_{j=1}^{S} |h^-_{M,j}|^2.$$

is a decomposition of h_M. Let also $F = [f_1, \ldots, f_{r'+s'}]$. Since M is also the multiplier of F, we necessarily have $\deg(F) = k$. Then, by our hypotheses,

$$h_M(z,\bar{z}) = \sum_{j=1}^{R} |h^+_{M,j}|^2 - \sum_{j=1}^{S} |h^-_{M,j}|^2 = \sum_{j=1}^{r'} |f_j|^2 - \sum_{j=r'+1}^{r'+s'} |f_j|^2.$$

Since the polynomials $h^+_{M,j}$ and $h^-_{M,j}$ are linearly independent by construction, it follows from Lemma 3.3 that $r' \geq R$, $s' \geq S$ and there exist degree-k homogeneous polynomials $\{\psi_1, \ldots, \psi_Q\} \subset \mathbb{C}[z_1, \ldots, z_{r+s}]$, where $Q \leq r' + s' - R - S$, and $W \in M(r'+s', R+S+Q; \mathbb{C})$ satisfying $\bar{W}^t H_{r',s'} W = H_{R,S,Q}$ such that

$$(f_1, \ldots, f_{r'+s'})^t = W(h^+_{M,1}, \ldots, h^+_{M,R}, h^-_{M,1}, \ldots, h^-_{M,S}, \psi_1, \ldots, \psi_Q)^t \tag{4}$$

Now, it is immediate that the map $\widetilde{F} : D_{r,s} \to D_{R,S,Q}$ defined by

$$\widetilde{F} = [h^+_{M,1}, \ldots, h^+_{M,R}, h^-_{M,1}, \ldots, h^-_{M,S}, \psi_1, \ldots, \psi_Q]$$

is a rational proper map. Furthermore, if we define $H : D_{R,S,Q} \to D_{r',s'}$ by

$$H([\zeta_1, \ldots, \zeta_{R+S+Q}]) = \left[\sum_{j=1}^{R+S+Q} w_{1,j}\zeta_j \ , \ \ldots \ , \sum_{j=1}^{R+S+Q} w_{r'+s',j}\zeta_j \right],$$

where $W = (w_{i,j})$, then H is a linear proper embedding and $F = H \circ \widetilde{F}$ from Eq (4). The proof is now complete. $\square$

4 Rational proper maps from $D_{2,2}$

The complex unit balls $\mathbb{B}^n \subset \mathbb{P}^n$ are special cases of generalized balls, and so are the complements of their closures, i.e. $\mathbb{P}^n \backslash \overline{\mathbb{B}^n}$. They are also precisely the generalized balls whose boundaries do not contain non-trivial complex analytic subvarieties. Other than these special cases, the simplest generalized ball is $D_{2,2} \subset \mathbb{P}^3$. Its boundary contains a family of $\mathbb{P}^1$ parametrized by the unitary group $U(2)$ [Ng2]. The existence of non-trivial compact subvarieties in the boundary is crucial to the study of generalized balls (see [Ng1, Ng2], for example). In the rest of this article, we will focus on the rational proper maps from $D_{2,2}$ to another generalized ball.

Let $r, s \geq 2$ and $F : D_{2,2} \dashrightarrow \mathbb{P}^{r+s-1} \supset D_{r,s}$ be a rational proper map. From the results of Baouendi-Huang [BH], it is known that F is equivalent to the standard linear embedding if $(r,s)=(2,2)$, $(2,3)$ or $(3,2)$. On the other hand, by considering the following simple example of a rational proper map from $D_{2,2}$ to $D_{3,3}$, given by

$$[z_1,z_2,z_3,z_4]\mapsto[z_1^2,\sqrt{2}z_1z_2,z_2^2,z_3^2,\sqrt{2}z_3z_4,z_4^2],$$

we see that similar rigidity does not hold for $r,s\geq 3$.

We first have the following observation for rational proper maps from $D_{2,2}$ with a fixed degree.

Proposition 4.1. *Let $F:D_{2,2}\dashrightarrow\mathbb{P}^{r'+s'-1}\supset D_{r',s'}$ be a rational proper map and $deg(F)=k\geq 2$. Then there exist unique integers $m,n,q\geq 0$, with $3\leq m\leq r'$, $3\leq n\leq s'$, $m+n\leq\binom{k+3}{3}$, $q\leq r'+s'-m-n$, and a degree-k rational proper map $F^\dagger:D_{2,2}\dashrightarrow\mathbb{P}^{m+n-1}\supset D_{m,n}$, with $F^\dagger=[f_1^\dagger,\ldots,f_{m+n}^\dagger]$, such that there is a factorization $F=H\circ\widetilde{F}$, where $H:D_{m,n,q}\to D_{r',s'}$ is a linear proper embedding and $\widetilde{F}:D_{2,2}\dashrightarrow\mathbb{P}^{m+n+q-1}\supset D_{m,n,q}$ is a rational proper map, with $\widetilde{F}=[f_1^\dagger,\ldots,f_{m+n}^\dagger,\psi_1,\ldots,\psi_q]$ for some degree-k homogeneous polynomials $\{\psi_1,\ldots,\psi_q\}\subset\mathbb{C}[z_1,z_2,z_3,z_4]$.*

Proof. Let $h_F=\|F\|^2_{r',s'}$. Then h_F is a real-valued bihomogeneous polynomial in $\mathbb{C}[z_1,\bar z_1,\ldots,z_4,\bar z_4]$ of degree (k,k). Thus, if we write $h_F=\sum_{|I|=|J|=k}a_{I\bar J}z^I\bar z^J$ using the multi-index notation, then the coefficients $a_{I\bar J}$ constitute an $N\times N$ Hermitian matrix, where $N=\binom{k+3}{3}$, which is the maximum number of linearly independent degree-k monomials in $\mathbb{C}[z_1,\ldots,z_4]$. Therefore, if (m,n) is the signature of h_F, then we must have $m+n\leq N$. Now by Theorem 3.6, we have $m\leq r'$, $n\leq s'$ and there exist a unique non-negative integer $q\leq r'+s'-m-n$, and a linear proper embedding $H:D_{m,n,q}\to D_{r',s'}$ such that $F=H\circ\widetilde{F}$, where $\widetilde{F}:D_{2,2}\dashrightarrow\mathbb{P}^{m+n+q-1}\supset D_{m,n,q}$ is a rational proper map, with $\widetilde{F}=[f_1^\dagger,\ldots,f_{m+n}^\dagger,\psi_1,\ldots,\psi_q]$, for some degree-$k$ homogeneous polynomials $\{\psi_1,\ldots,\psi_q\}\subset\mathbb{C}[z_1,z_2,z_3,z_4]$ and $F^\dagger:=[f_1^\dagger,\ldots,f_{m+n}^\dagger]$ is a canonical rational map associated to $h_F/\|z\|^2_{2,2}$. Finally, since $\deg(F)\geq 2$, we have $m,n\geq 3$ since otherwise F is necessarily of degree 1, as mentioned above. □

4.1 The degree-2 case

In this section, we are going to look closely at the degree-2 rational proper maps from $D_{2,2}$ to an arbitrary generalized ball. Let $F:D_{2,2}\dashrightarrow\mathbb{P}^{r+s-1}\supset D_{r,s}$ be a degree-2 rational proper map. By Proposition 3.1, there corresponds a real-valued bihomogeneous polynomial $M\in\mathbb{C}[z_1,\bar z_1,\ldots,z_4,\bar z_4]$ of bidegree $(1,1)$. Conversely, by Theorem 3.6, any rational proper map from $D_{2,2}$ to another generalized ball having the same multiplier M, is essentially a canonical rational proper map associated to M, up to trivial modifications. Thus, together with Proposition 4.1, we may, for the purpose of studying degree-2 rational proper maps from $D_{2,2}$, restrict ourselves to the cases where $s \geq r \geq 3$, with $r + s \leq \binom{5}{3} = 10$.

Recall that a canonical rational proper map associated to M is obtained by computing a decomposition for

$$\begin{aligned} h_M(z,\bar{z}) &:= (|z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2) M(z,\bar{z}) \\ &= \sum_{j=1}^{r} |h^+_{M,j}|^2 - \sum_{j=1}^{s} |h^-_{M,j}|^2, \end{aligned}$$

where $h^+_{M,j}$, $h^-_{M,j}$ are linearly independent homogeneous polynomials of degree 2 in $\mathbb{C}[z_1, z_1, z_3, z_4]$.

Now let $M(z,\bar{z}) = (\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4) A_M (z_1, z_2, z_3, z_4)^t \in \mathbb{C}[z_1, \bar{z}_1, \ldots, z_4, \bar{z}_4]$ be a multiplier on $D_{2,2}$, where $A_M = (a_{ij}) \in M(4,4;\mathbb{C})$ is a Hermitian matrix. We also write $\mathbf{z} = (z_1, z_2, z_3, z_4)^t$ and $\mathbf{Z} = (z_1^2, z_1 z_2, z_1 z_3, \ldots, z_4^2)^t$, in which elements are arranged in the lexicographical order. Then,

$$\begin{aligned} h_M(z,\bar{z}) &= (|z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2) M(z,\bar{z}) \\ &= \bar{\mathbf{z}}^t \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \mathbf{z}\, M(z,\bar{z}) \\ &= \bar{\mathbf{z}}^t \begin{pmatrix} M(z,\bar{z}) & 0 & 0 & 0 \\ 0 & M(z,\bar{z}) & 0 & 0 \\ 0 & 0 & -M(z,\bar{z}) & 0 \\ 0 & 0 & 0 & -M(z,\bar{z}) \end{pmatrix} \mathbf{z} \\ &= \bar{\mathbf{z}}^t \begin{pmatrix} \bar{\mathbf{z}}^t A_M \mathbf{z} & 0 & 0 & 0 \\ 0 & \bar{\mathbf{z}}^t A_M \mathbf{z} & 0 & 0 \\ 0 & 0 & -\bar{\mathbf{z}}^t A_M \mathbf{z} & 0 \\ 0 & 0 & 0 & -\bar{\mathbf{z}}^t A_M \mathbf{z} \end{pmatrix} \mathbf{z} \\ &= \bar{\mathbf{z}}^t \begin{pmatrix} \bar{\mathbf{z}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{z}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{z}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \bar{\mathbf{z}} \end{pmatrix}^t \begin{pmatrix} A_M & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_M & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -A_M & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -A_M \end{pmatrix} \begin{pmatrix} \mathbf{z} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{z} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{z} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{z} \end{pmatrix} \mathbf{z} \\ &= \begin{pmatrix} \bar{z}_1 \bar{\mathbf{z}} & \bar{z}_2 \bar{\mathbf{z}} & \bar{z}_3 \bar{\mathbf{z}} & \bar{z}_4 \bar{\mathbf{z}} \end{pmatrix} \begin{pmatrix} A_M & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_M & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -A_M & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -A_M \end{pmatrix} \begin{pmatrix} z_1 \mathbf{z} \\ z_2 \mathbf{z} \\ z_3 \mathbf{z} \\ z_4 \mathbf{z} \end{pmatrix} \\ &= \bar{\mathbf{Z}}^t B_M \mathbf{Z}, \end{aligned}$$

where

$$B_M = \begin{pmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{21} & a_{11}+a_{22} & a_{23} & a_{24} & a_{12} & a_{13} & a_{14} & 0 & 0 & 0 \\
a_{31} & a_{32} & a_{33}-a_{11} & a_{34} & 0 & -a_{12} & 0 & -a_{13} & -a_{14} & 0 \\
a_{41} & a_{42} & a_{43} & a_{44}-a_{11} & 0 & 0 & -a_{12} & 0 & -a_{13} & -a_{14} \\
 & & & & & & & & & \\
0 & a_{21} & 0 & 0 & a_{22} & a_{23} & a_{24} & 0 & 0 & 0 \\
0 & a_{31} & -a_{21} & 0 & a_{32} & a_{33}-a_{22} & a_{34} & -a_{23} & -a_{24} & 0 \\
0 & a_{41} & 0 & -a_{21} & a_{42} & a_{43} & a_{44}-a_{22} & 0 & -a_{23} & -a_{24} \\
 & & & & & & & & & \\
0 & 0 & -a_{31} & 0 & 0 & -a_{32} & 0 & -a_{33} & -a_{34} & 0 \\
0 & 0 & -a_{41} & -a_{31} & 0 & -a_{42} & -a_{32} & -a_{43} & -a_{33}-a_{44} & -a_{34} \\
0 & 0 & 0 & -a_{41} & 0 & 0 & -a_{42} & 0 & -a_{43} & -a_{44}
\end{pmatrix}$$

The signature of B_M (as a Hermitian matrix) will determine the generalized ball $D_{r,s}$ to which $D_{2,2}$ is mapped by a canonical rational proper map F_M associated to M. Furthermore, a maximal linearly independent set of eigenvectors of the non-zero eigenvalues of B_M give precisely the coefficients (with respect to the basis $\mathbf{Z}$) of the component functions of a canonical rational proper map associated to M. It turns out that by considering only multipliers of the diagonal type, we can obtain degree-2 canonical rational proper maps $D_{2,2} \dashrightarrow \mathbb{P}^{r+s-1} \supset D_{r,s}$ for all possible pairs (r,s) (i.e. $r+s \leq 10$, as given by Proposition 4.1). These examples are listed as follows.

$M(z,\bar{z})$	(r,s)	$M(z,\bar{z})$	(r,s)
$\lvert z_1\rvert^2+\lvert z_2\rvert^2+\lvert z_3\rvert^2+\lvert z_4\rvert^2$	(3,3)	$2\lvert z_1\rvert^2+\lvert z_2\rvert^2+2\lvert z_3\rvert^2+\lvert z_4\rvert^2$	(4,4)
$\lvert z_1\rvert^2+\lvert z_2\rvert^2$	(3,4)	$3\lvert z_1\rvert^2+\lvert z_2\rvert^2+2\lvert z_3\rvert^2+\lvert z_4\rvert^2$	(4,5)
$2\lvert z_1\rvert^2+\lvert z_2\rvert^2+\lvert z_3\rvert^2+\lvert z_4\rvert^2$	(3,5)	$4\lvert z_1\rvert^2+2\lvert z_2\rvert^2+3\lvert z_3\rvert^2+\lvert z_4\rvert^2$	(4,6)
$3\lvert z_1\rvert^2+2\lvert z_2\rvert^2+2\lvert z_3\rvert^2+\lvert z_4\rvert^2$	(3,6)	$2\lvert z_1\rvert^2+2\lvert z_2\rvert^2+3\lvert z_3\rvert^2+\lvert z_4\rvert^2$	(5,5)
$2\lvert z_1\rvert^2+2\lvert z_2\rvert^2+\lvert z_3\rvert^2+\lvert z_4\rvert^2$	(3,7)		

Note that the canonical rational proper maps associated to the multipliers given in the above table are actually *holomorphic* proper maps. The holomorphicity follows from the fact that the above multipliers are all strictly positive on $D_{2,2}$ and hence the indeterminacies of the rational maps are outside $D_{2,2}$.

Consider the canonical left action of $U(2,2)$ on $D_{2,2}$ given by matrix multiplication, i.e. for $V \in U(2,2)$, the point $[z_1,z_2,z_3,z_4] \in D_{2,2}$ is mapped to $[z_1,z_2,z_3,z_4]V^t$. Then $U(2,2)$ also acts on the set of multipliers of bi-degree $(1,1)$ naturally,

$$\begin{aligned}
M(z,\bar{z}) &= (\bar{z}_1,\bar{z}_2,\bar{z}_3,\bar{z}_4)A_M(z_1,z_2,z_3,z_4)^t \\
\mapsto M^{(V)}(z,\bar{z}) &:= (\bar{z}_1,\bar{z}_2,\bar{z}_3,\bar{z}_4)\overline{V}^t A_M V(z_1,z_2,z_3,z_4)^t, \qquad (5)
\end{aligned}$$

for $V \in U(2,2)$. That is, $A_{M^{(V)}} = \overline{V}^t A_M V$. Conversely, it is clear that the canonical rational proper maps given rise by $M(z,\bar{z})$ and $M^{(V)}(z,\bar{z})$ are equivalent up to actions of $U(2,2)$ on $D_{2,2}$.

For the maximal rank case, we have the the following:

Theorem 4.2. *For each of the generalized balls $D_{r,s}$ with $r,s \geq 3$ and $r+s=10$, there exists a real-parameter family of degree-2 proper holomorphic maps $F_t : D_{2,2} \to D_{r,s}$, such that F_t and $F_{t'}$ are not equivalent if $t \neq t'$.*

Proof. Fix a generalized ball $D_{r,s}$ with $r,s \geq 3$ and $r+s=10$. From the above table, we see that there is a holomorphic proper map $F_0 : D_{2,2} \to D_{r,s}$ whose multiplier M_0 is given by a positive-definite diagonal matrix. Recall that, if we let $h_{F_0}(z,\bar{z}) = \|F_0\|^2_{r,s}$, then $h_{F_0} = \|z\|^2_{2,2} M_0(z,\bar{z})$.

Write $M_0(z,\bar{z}) = (\bar{z}_1,\bar{z}_2,\bar{z}_3,\bar{z}_4)A_0(z_1,z_2,z_3,z_4)^t$, where $A_0 \in M(4,4;\mathbb{C})$ is a real diagonal matrix. Denote by $H(4) \subset M(4,4;\mathbb{C})$ the set of 4×4 Hermitian matrices. Since h_{F_0} is of maximal rank, there is a neighborhood $\mathcal{U}$ of A_0 in $H(4)$ such that for every $A \in \mathcal{U}$, the function $h_A(z,\bar{z}) := \|z\|^2_{2,2} M_A(z,\bar{z})$ is also of the same signature as h_{F_0}. (Here, $M_A(z,\bar{z}) = (\bar{z}_1,\bar{z}_2,\bar{z}_3,\bar{z}_4)A(z_1,z_2,z_3,z_4)^t$.) Furthermore, as $M_0(z,\bar{z})$ is strictly positive on $\mathbb{P}^3$, by shrinking $\mathcal{U}$ if necessary, every $A \in \mathcal{U}$ will give rise to a multiplier $M_A(z,\bar{z})$ which remains strictly positive on $\mathbb{P}^3$ and thus the canonical rational proper maps associated to $h_A(z,\bar{z})$ are also holomorphic, for every $A \in \mathcal{U}$.

Consider now the canonical left action of $U(2,2)$ on $D_{2,2}$ and the induced action on the set of multipliers of bi-degree $(1,1)$, as given in Eq (5). Let $E \subset U(2,2)$ be the subgroup consisting of diagonal matrices. Since A_0 is diagonal, we see that E belongs to the isotropy subgroup of $U(2,2)$ fixing M_0. Now $\dim_{\mathbb{R}}(U(2,2)) = 16$ and $\dim_{\mathbb{R}}(E) = 4$, but $\dim_{\mathbb{R}}(\mathcal{U}) = \dim_{\mathbb{R}}(H(4)) = 16$, we thus see that there exists a real parameter family $\{A_t : t \in (-1,1)\} \subset \mathcal{U}$ such that A_t and $A_{t'}$ are not equivalent under the action $U(2,2)$ whenever $t \neq t'$. Then $\{A_t : t \in (-1,1)\}$ gives a family of holomorphic proper maps from $D_{2,2}$ to $D_{r,s}$ which are pairwise non-equivalent and the proof is complete. $\square$

4.1.1 Complete determination of multipliers for the minimal case

We will now determine (up to automorphisms) all the multipliers whose canonical rational proper maps are of degree 2 and are from $D_{2,2}$ to $D_{3,3}$. To start with, we do certain normalization for A_M to simplify the problem, as follows. As before, we write the multiplier $M(z,\bar{z}) = (\bar{z}_1,\bar{z}_2,\bar{z}_3,\bar{z}_4)A_M(z_1,z_2,z_3,z_4)^t$, where $A_M = (a_{ij}) \in M(4,4;\mathbb{C})$.

Given a rational proper map F from $D_{2,2} \dashrightarrow \mathbb{P}^{r+s-1} \supset D_{r,s}$, since the set of indeterminacy of F is of co-dimension at least 2, a general line sitting inside $D_{2,2}$ does not intersect the set of indeterminacy. We may assume that $L = \{[z_1,z_2,0,0] \subset D_{2,2} : [z_1,z_2] \in \mathbb{P}^1\}$ is one of such lines. Moreover, by composing with some automorphism, we may further assume that $\|F([1,0,0,0])\|^2_{r,s} \neq 0$, $\|F([0,1,0,0])\|^2_{r,s} \neq 0$ since the zero set of $\|F\|^2_{r,s}$ is at least of real co-dimension 1 in $\mathbb{P}^3$. The condition $\|F([1,0,0,0])\|^2_{r,s} \neq 0$ gives $a_{11} \neq 0$ and $\|F([0,1,0,0])\|^2_{r,s} \neq 0$ gives $a_{22} \neq 0$.

Recall the canonical left action of $U(2,2)$ on $D_{2,2}$ and the induced action on the set of multipliers of bi-degree $(1,1)$, as given in Eq. (5). Since $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $\begin{bmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{bmatrix}$ are Hermitian, we can choose an element $U = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \in U(2) \times U(2) \subset U(2,2)$ such that both

$$\overline{U}_1^t \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} U_1 \quad \text{and} \quad \overline{U}_2^t \begin{bmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{bmatrix} U_2$$

are diagonal matrices. Note that such an automorphism on $D_{2,2}$ fixes the line L and therefore, combining with the previous paragraph, we may now take

$$a_{11} \neq 0, \quad a_{22} \neq 0 \quad \text{and} \quad a_{12} = a_{21} = a_{34} = a_{43} = 0. \tag{6}$$

Under this normalization, we will then determine all possible A_M (and hence all multipliers $M(z,\bar{z})$) whose canonical rational proper maps are from $D_{2,2}$ to $D_{3,3}$. The computation details are given in Section 5, where the solutions are computed in different cases. We observe that some cases there can be combined and we list the final (combined) solutions here.

There are three types.

<u>**Type A**</u>

$$\begin{pmatrix} a & 0 & \sqrt{ack}e^{i\theta_{13}} & \sqrt{ad\ell}e^{i\theta_{14}} \\ 0 & b & \sqrt{bc\ell}e^{i\theta_{23}} & \sqrt{bdk}e^{i\theta_{24}} \\ \sqrt{ack}e^{-i\theta_{13}} & \sqrt{bc\ell}e^{-i\theta_{23}} & c & 0 \\ \sqrt{ad\ell}e^{-i\theta_{14}} & \sqrt{bdk}e^{-i\theta_{24}} & 0 & d \end{pmatrix},$$

where $a,b,c,d,k,\ell,\theta_{13},\theta_{14},\theta_{23},\theta_{24} \in \mathbb{R}$, such that $ab \neq 0$, $ack \geq 0$, $ad\ell \geq 0$, $bc\ell \geq 0$, and either

$$a = b = c = d \quad \text{or} \quad \begin{cases} a+b = c+d \\ k + \ell = 1. \end{cases}$$

Furthermore,

$$\theta_{13} + \theta_{24} = \theta_{14} + \theta_{23} + \pi.$$

Here we have combined the solutions for **Sub-case I.1**, **Sub-case II.1**, **Sub-case II.3**, the first two solutions in **Case IV** computed in Section 5.

<u>**Type B**</u>

$$\begin{pmatrix} a & 0 & re^{i\theta_{13}} & re^{i\theta_{14}} \\ 0 & -a & re^{i\theta_{23}} & re^{i\theta_{24}} \\ re^{-i\theta_{13}} & re^{-i\theta_{23}} & 0 & 0 \\ re^{-i\theta_{14}} & re^{-i\theta_{24}} & 0 & 0 \end{pmatrix},$$

where $a, r \in \mathbb{R}$, with $a \neq 0$, and $\theta_{13}, \theta_{14}, \theta_{23}, \theta_{24} \in \mathbb{R}$ satisfying

$$\theta_{13} + \theta_{24} = \theta_{14} + \theta_{23}.$$

Here we have combined the solutions for **Sub-case I.2** and the last solution of **Case IV** computed in Section 5.

Type C

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & re^{i\theta} \\ 0 & 0 & c & 0 \\ 0 & re^{-i\theta} & 0 & d \end{pmatrix},$$

where $a, b, c, d, r, \theta \in \mathbb{R}$, such that $ab \neq 0$ and

$$\begin{cases} a+b &= c+d \\ c(a-c) &= 0 \\ bd-ac &= r^2. \end{cases}$$

Here we have combined the two kinds of solutions for **Case III** computed in Section 5.

4.1.2 Determination of all degree-2 rational proper maps from $D_{2,2}$ to $D_{3,3}$

We can now determine all degree-2 rational proper maps from $D_{2,2}$ to $D_{3,3}$.

Theorem 4.3. *Let $F : D_{2,2} \dashrightarrow \mathbb{P}^5 \supset D_{3,3}$ be a degree-2 rational proper map. Then F is equivalent to either (i) a canonical rational proper map associated to a multiplier of Type A, Type B, or Type C; or (ii) a rational map of the form*

$$[z_1, z_2, z_3, z_4] \mapsto [z_1\varphi, z_2\varphi, \psi, z_3\varphi, z_4\varphi, \psi],$$

where $\varphi, \psi \in \mathbb{C}[z_1, z_2, z_3, z_4]$ with $\deg(\varphi) = 1$ and $\deg(\psi) = 2$.

Proof. Let $h_F = \|F\|^2_{3,3}$ and let its signature be (r, s). Then we have $r + s \geq 4$ since otherwise a decomposition of h_F as in Eq (2) of Section 3.2 would give a rational proper map from $D_{2,2}$ to $D_{r,s}$ with $r + s \leq 3$, which does not exist.

If $r + s = 4$, then we necessarily have $r = s = 2$ and any canonical rational proper map $F^\dagger$ associated to the multiplier $h_F/\|z\|^2_{2,2}$ is from $D_{2,2}$ to $D_{2,2}$. On the other hand, it is known that any local proper holomorphic map from $D_{2,2}$ to $D_{2,2}$ is the restriction of an automorphism of $D_{2,2}$ by Baouendi-Huang [BH]. But $\deg(F^\dagger) = 2$ and thus the only possibility is that, up to an automorphism of $D_{2,2}$, we have

$$F^\dagger = [z_1\varphi, z_2\varphi, z_3\varphi, z_4\varphi],$$

for some degree-1 polynomial $\varphi \in \mathbb{C}[z_1, z_2, z_3, z_4]$. Now by Theorem 3.6, there is a non-negative integer $Q \leq 2$ and a linear proper embedding $H : D_{2,2,Q} \to D_{3,3}$ such that $F = H \circ \widetilde{F}$, where $\widetilde{F} : D_{2,2} \dashrightarrow \mathbb{P}^{Q+3} \supset D_{2,2,Q}$ is a rational proper map with $\widetilde{F} = [z_1\varphi, z_2\varphi, z_3\varphi, z_4\varphi, \psi_1, \psi_Q]$, for some degree-2 polynomials $\{\psi_1, \psi_Q\} \subset \mathbb{C}[z_1, z_2, z_3, z_4]$. (In case $Q = 0$, we have $\widetilde{F} = [z_1\varphi, z_2\varphi, z_3\varphi, z_4\varphi]$.) Note that the existence of the linear proper embedding H is equivalent to the existence of a matrix $W \in M(6, Q+4; \mathbb{C})$ of full rank such that $\bar{W}^t H_{3,3} W = H_{2,2,Q}$. We thus deduce that $Q = 2$ is not possible and hence $Q = 0$ or $Q = 1$.

When $Q = 0$, the component functions of $\widetilde{F}$ are not relatively prime and the "true" degree of $\widetilde{F}$ and F is actually 1 and therefore should be neglected. When $Q = 1$, by composing with an autormophism of $D_{3,3}$ if necessary, we can always assume that

$$W = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus,

$$F = H \circ \widetilde{F} = [z_1\varphi, z_2\varphi, z_3\varphi, z_4\varphi, \psi_1] \cdot W^t = [z_1\varphi, z_2\varphi, \psi_1, z_3\varphi, z_4\varphi, \psi_1].$$

If $r+s = 5$, then any canonical rational proper map $F^\dagger$ associated to the multiplier $h_F/\|z\|^2_{2,2}$ is from $D_{2,2}$ to $D_{2,3}$ (or $D_{3,2}$). And again by Baouendi-Huang [BH], any local proper holomorphic map from $D_{2,2}$ to $D_{2,3}$ (or $D_{3,2}$) is a restriction of a linear proper embedding and therefore up to automorphisms, $F^\dagger$ must be proportional to $F_0 := [z_1, z_2, z_3, z_4, 0]$ (or $F_0' := [z_1, z_2, 0, z_3, z_4]$). This contradicts the fact that h_F is of signature $(2, 3)$ (or $(3, 2)$).

Finally, if $r + s = 6$, then using a similar argument as above, it is easy to see that we must have $(r, s) = (3, 3)$ and hence F is a canonical rational proper map associated to a multiplier of Type A, Type B or Type C. □

5 Appendix

In this section, we provide the details for solving all possible Hermitian matrices

$$A = \begin{pmatrix} a_{11} & 0 & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & 0 & a_{44} \end{pmatrix},$$

where $a_{11} \neq 0$, $a_{22} \neq 0$ (c.f. Eq. (6)), such that the corresponding multiplier $M_A(z, \bar{z}) := (\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4)A(z_1, z_2, z_3, z_4)^t$ gives rise to canonical rational proper maps from $D_{2,2}$ to $D_{3,3}$.

Let $h_A(z,\bar{z}) := \|z\|^2_{2,2} M_A(z,\bar{z}) = \bar{\mathbf{Z}}^t B_A \mathbf{Z}$, where

$$B_A = \begin{pmatrix} a_{11} & 0 & a_{13} & a_{14} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{11}+a_{22} & a_{23} & a_{24} & 0 & a_{13} & a_{14} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33}-a_{11} & 0 & 0 & 0 & 0 & -a_{13} & -a_{14} & 0 \\ a_{41} & a_{42} & 0 & a_{44}-a_{11} & 0 & 0 & 0 & 0 & -a_{13} & -a_{14} \\ \\ 0 & 0 & 0 & 0 & a_{22} & a_{23} & a_{24} & 0 & 0 & 0 \\ 0 & a_{31} & 0 & 0 & a_{32} & a_{33}-a_{22} & 0 & -a_{23} & -a_{24} & 0 \\ 0 & a_{41} & 0 & 0 & a_{42} & 0 & a_{44}-a_{22} & 0 & -a_{23} & -a_{24} \\ \\ 0 & 0 & -a_{31} & 0 & 0 & -a_{32} & 0 & -a_{33} & 0 & 0 \\ 0 & 0 & -a_{41} & -a_{31} & 0 & -a_{42} & -a_{32} & 0 & -a_{33}-a_{44} & 0 \\ 0 & 0 & 0 & -a_{41} & 0 & 0 & -a_{42} & 0 & 0 & -a_{44} \end{pmatrix}.$$

and $\mathbf{Z} = (z_1^2, z_1z_2, z_1z_3, \ldots, z_4^2)^t$, in which elements are arranged in the lexicographical order.

Recall that the if $M_A(z,\bar{z})$ is a multiplier on $D_{2,2}$ and the signature of B_A (which is the same as the signature of h_A) is (r,s), then the canonical rational maps associated to M_A are from $D_{2,2}$ to $D_{r,s}$. Since we are looking for canonical rational maps from $D_{2,2}$ to $D_{3,3}$, we have $\operatorname{rank}(B_A) = 3+3 = 6$. In addition, it is known that (i) there are no local proper holomorphic map from $D_{2,2}$ to $D_{r,s}$ if $r=1$ or $s=1$; and (ii) every local proper holomorphic map from $D_{2,2}$ to $D_{2,4}$ or $D_{4,2}$ is equivalent to a linear map and hence of degree 1. Therefore, it suffices to determine all matrices A such that $\operatorname{rank}(B_A) = 6$.

Remark. There is actually an additional condition on A for M_A to be a multiplier on $D_{2,2}$, namely, there exists a connected open subset $\mathcal{U} \subset \mathbb{P}^3$ with $\mathcal{U} \cap \partial D_{2,2} \neq \emptyset$ such that $M_A > 0$ on $\mathcal{U} \cap D_{2,2}$. We note here that by replacing A by $-A$ if necessary, such condition is always satisfied. Moreover, if B_A is of signature $(3,3)$, then the signature of B_{-A} is again $(3,3)$. Therefore, for the purpose of determining all degree-2 rational maps from $D_{2,2}$ to $D_{3,3}$, we just need to determine all matrices A such that $\operatorname{rank}(B_A) = 6$.

For later convenience, we let $B_A = (\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_{10}})$.

We will divide the problem into cases according the number of zero elements in the matrix $\begin{pmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{pmatrix} = \begin{pmatrix} \overline{a_{31}} & \overline{a_{41}} \\ \overline{a_{32}} & \overline{a_{24}} \end{pmatrix}$, as follows:

Case I: There is at most one element equal to zero.

Case II: There are two elements equal to zero.

Case III: There are three elements equal to zero.

Case IV: There are four elements equal to zero.

We will provide most of the calculation detail for **Case I** and **Case II** since these are the cases with most non-zero coefficients and hence are the most difficult. For other cases, the calculations are along the same line and we will simply state the solutions.

5.1 Case I

By exploiting the symmetry $(z_1, z_2, z_3, z_4) \mapsto (z_1, z_2, z_4, z_3)$ of $D_{2,2}$, we may assume that

$$a_{13} \neq 0, \quad a_{24} \neq 0.$$

Thus, we also have

$$a_{31} = \overline{a_{13}} \neq 0, \quad a_{42} = \overline{a_{24}} \neq 0.$$

We further divide it into two sub-cases.

5.1.1 Sub-case I.1: $a_{11} + a_{22} \neq 0$

We choose the following six columns of B_A:

$$(\mathbf{b_1}, \mathbf{b_3}, \mathbf{b_9}, \mathbf{b_5}, \mathbf{b_2}, \mathbf{b_{10}})$$

$$= \begin{pmatrix} a_{11} & a_{13} & 0 & 0 & 0 & 0 \\ 0 & a_{23} & 0 & 0 & a_{11} + a_{22} & 0 \\ a_{31} & a_{33} - a_{11} & -a_{14} & 0 & a_{32} & 0 \\ a_{41} & 0 & -a_{13} & 0 & a_{42} & -a_{14} \\ & & & & & \\ 0 & 0 & 0 & a_{22} & 0 & 0 \\ 0 & 0 & -a_{24} & a_{32} & a_{31} & 0 \\ 0 & 0 & -a_{23} & a_{42} & a_{41} & -a_{24} \\ & & & & & \\ 0 & -a_{31} & 0 & 0 & 0 & 0 \\ 0 & -a_{41} & -a_{33} - a_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -a_{44} \end{pmatrix}$$

These six columns are linearly independent, which can be seen as follows. Obviously $\operatorname{rank}(\mathbf{b_5}, \mathbf{b_2}, \mathbf{b_{10}}) = 3$ since $a_{22}a_{31}a_{24} \neq 0$. Then $\operatorname{rank}(\mathbf{b_1}, \mathbf{b_3}, \mathbf{b_5}, \mathbf{b_2}, \mathbf{b_{10}}) = 5$ since $a_{11}a_{31} \neq 0$. Finally, if

$$\mathbf{b_9} = \beta_1 \mathbf{b_1} + \beta_3 \mathbf{b_3} + \beta_5 \mathbf{b_5} + \beta_2 \mathbf{b_2} + \beta_{10} \mathbf{b_{10}},$$

then $\beta_3 = 0$ (as $a_{31} \neq 0$) $\Rightarrow \beta_1 = 0$ (as $a_{11} \neq 0$) $\Rightarrow \beta_5 = 0$ (as $a_{22} \neq 0$) $\Rightarrow \beta_2 = 0$ (as $a_{11} + a_{22} \neq 0$) $\Rightarrow \mathbf{b_9} = \beta_{10}\mathbf{b_{10}}$ and we arrive at a contradiction as $a_{24} \neq 0$.

The condition $\operatorname{rank}(B_A) = 6$ now implies $\{\mathbf{b_4}, \mathbf{b_7}, \mathbf{b_8}, \mathbf{b_6}\} \subset \operatorname{Span}\{\mathbf{b_1}, \mathbf{b_3}, \mathbf{b_9}, \mathbf{b_5}, \mathbf{b_2}, \mathbf{b_{10}}\}$.

Thus,

$$\begin{pmatrix} a_{14} & 0 & 0 & 0 \\ a_{24} & a_{14} & 0 & a_{13} \\ 0 & 0 & -a_{13} & 0 \\ a_{44}-a_{11} & 0 & 0 & 0 \\ \\ 0 & a_{24} & 0 & a_{23} \\ 0 & 0 & -a_{23} & a_{33}-a_{22} \\ 0 & a_{44}-a_{22} & 0 & 0 \\ \\ 0 & 0 & -a_{33} & -a_{32} \\ -a_{31} & -a_{32} & 0 & -a_{42} \\ -a_{41} & -a_{42} & 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{13} & 0 & 0 & 0 & 0 \\ 0 & a_{23} & 0 & 0 & a_{11}+a_{22} & 0 \\ a_{31} & a_{33}-a_{11} & -a_{14} & 0 & a_{32} & 0 \\ a_{41} & 0 & -a_{13} & 0 & a_{42} & -a_{14} \\ \\ 0 & 0 & 0 & a_{22} & 0 & 0 \\ 0 & 0 & -a_{24} & a_{32} & a_{31} & 0 \\ 0 & 0 & -a_{23} & a_{42} & a_{41} & -a_{24} \\ \\ 0 & -a_{31} & 0 & 0 & 0 & 0 \\ 0 & -a_{41} & -a_{33}-a_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -a_{44} \end{pmatrix} \mathbf{X},$$

for some unique matrix $\mathbf{X} = \begin{pmatrix} \gamma_1 & \delta_1 & \mu_1 & \nu_1 \\ \gamma_2 & \delta_2 & \mu_2 & \nu_2 \\ \gamma_3 & \delta_3 & \mu_3 & \nu_3 \\ \gamma_4 & \delta_4 & \mu_4 & \nu_4 \\ \gamma_5 & \delta_5 & \mu_5 & \nu_5 \\ \gamma_6 & \delta_6 & \mu_6 & \nu_6 \end{pmatrix} \in M(6,4;\mathbb{C})$.

Given a column on the left-hand side, say the first column, one can choose six rows on the right-hand side to solve for $(\gamma_1, \dots, \gamma_6)$. Then the remaining four rows will give compatibility equations in a_{ij}. The set of equations obtained in this way for each column on the left-hand side will then be solved to obtain the solution for a_{ij}.

Lemma 5.1. $a_{44} \neq 0$

Proof. The last row implies the lemma since $a_{42} \neq 0$. □

First column

$$\begin{array}{llllll} & \text{5th row gives} & \gamma_4 = 0 & \Longrightarrow & \text{8th row gives} & \gamma_2 = 0 \\ \Longrightarrow & \text{1st row gives} & \gamma_1 = \dfrac{a_{14}}{a_{11}} & \Longrightarrow & \text{10th row gives} & \gamma_6 = \dfrac{a_{41}}{a_{44}} \\ \Longrightarrow & \text{2th row gives} & \gamma_5 = \dfrac{a_{24}}{a_{11}+a_{22}} & \Longrightarrow & \text{9th row gives} & \gamma_3 = \dfrac{a_{31}}{a_{33}+a_{44}} \end{array}$$

Compatibility equations

6th row:

$$-a_{24}\gamma_3 + a_{31}\gamma_5 = 0 \quad \Leftrightarrow \quad a_{11} + a_{22} = a_{33} + a_{44}$$

3rd row:

$$a_{31}\gamma_1 - a_{14}\gamma_3 + a_{32}\gamma_5 = 0 \quad \Leftrightarrow \quad -a_{11}a_{24}a_{32} = a_{22}a_{14}a_{31}$$

7th row:

$$-a_{23}\gamma_3 + a_{41}\gamma_5 - a_{24}\gamma_6 = 0 \quad \Leftrightarrow \quad -a_{33}a_{24}a_{41} = a_{44}a_{23}a_{31}$$

4th row:

$$a_{44} - a_{11} = a_{41}\gamma_1 - a_{13}\gamma_3 + a_{42}\gamma_5 - a_{14}\gamma_6 \quad \Leftrightarrow \quad (a_{11} - a_{44})\left(1 - \frac{|a_{14}|^2}{a_{11}a_{44}}\right) = \frac{|a_{13}|^2 - |a_{24}|^2}{a_{11} + a_{22}}$$

These equations also imply

Lemma 5.2. $a_{33} \neq 0$.

Proof. If $a_{33} = 0$, the equation from 7th row gives $a_{23} = 0$. But then the equation from 3rd row gives $a_{14} = 0$. We get a contradiction since we are in **Case I**. □

Second column

The equations are the same as those obtained in first column after switching $1 \leftrightarrow 2$ in the subscripts of a_{ij}. Thus, there is only one new equation here:

Compatibility equation

$$(a_{22} - a_{44})\left(1 - \frac{|a_{24}|^2}{a_{22}a_{44}}\right) = \frac{|a_{23}|^2 - |a_{14}|^2}{a_{11} + a_{22}}$$

Third column

$$\begin{array}{llll} & \text{5th row gives} \quad \mu_4 = 0 & \Longrightarrow & \text{10th row gives} \quad \mu_6 = 0 \\ \Longrightarrow & \text{8th row gives} \quad \mu_2 = \dfrac{a_{33}}{a_{31}} & \Longrightarrow & \text{1st row gives} \quad \mu_1 = \dfrac{-a_{13}a_{33}}{a_{11}a_{31}} \\ \Longrightarrow & \text{9th row gives} \quad \mu_3 = \dfrac{-a_{33}a_{41}}{a_{31}(a_{33} + a_{44})} & \Longrightarrow & \text{2nd row gives} \quad \mu_5 = \dfrac{-a_{23}a_{33}}{a_{31}(a_{11} + a_{22})} \end{array}$$

Compatibility equations

(We will take into account $a_{11} + a_{22} = a_{33} + a_{44}$, and $a_{33} \neq 0$, which are obtained earlier.)

4th row:

$$a_{41}\mu_1 - a_{13}\mu_3 + a_{42}\mu_5 = 0 \quad \Leftrightarrow \quad -a_{11}a_{23}a_{42} = a_{22}a_{13}a_{41}$$

6th row:

$$-a_{23} = -a_{24}\mu_3 + a_{31}\mu_5 \quad \Leftrightarrow \quad -a_{33}a_{24}a_{41} = a_{44}a_{23}a_{31}$$

7th row:

$$-a_{23}\mu_3 + a_{41}\mu_5 = 0 \quad \Leftrightarrow \quad 0 = 0$$

3rd row:

$$-a_{13} = a_{31}\mu_1 + (a_{33} - a_{11})\mu_2 - a_{14}\mu_3 + a_{32}\mu_5 \quad \Leftrightarrow \quad (a_{11} - a_{33})\left(1 - \frac{|a_{13}|^2}{a_{11}a_{33}}\right) = \frac{|a_{14}|^2 - |a_{23}|^2}{a_{11} + a_{22}}$$

Fourth column

$$\begin{array}{lllllll} & \text{10th row gives} & \nu_6 = 0 & \Longrightarrow & \text{8th row gives} & \nu_2 = \dfrac{a_{32}}{a_{31}} \\ \Longrightarrow & \text{5th row gives} & \nu_4 = \dfrac{a_{23}}{a_{22}} & \Longrightarrow & \text{1st row gives} & \nu_1 = \dfrac{-a_{13}a_{32}}{a_{31}a_{11}} \\ \Longrightarrow & \text{2nd row gives} & \nu_5 = \dfrac{|a_{13}|^2 - |a_{23}|^2}{a_{31}(a_{11} + a_{22})} & \Longrightarrow & \text{9th row gives} & \nu_3 = \dfrac{a_{31}a_{42} - a_{32}a_{41}}{a_{31}(a_{33} + a_{44})} \end{array}$$

Compatibility equations

(We will take into account $a_{11} + a_{22} = a_{33} + a_{44}$, and $a_{33} \neq 0$, which are obtained earlier.)

3rd row:

$$a_{31}\nu_1 + (a_{33} - a_{11})\nu_2 - a_{14}\nu_3 + a_{32}\nu_5 = 0 \quad \Leftrightarrow \quad (a_{11} - a_{33})\left(1 - \frac{|a_{13}|^2}{a_{11}a_{33}}\right) = \frac{|a_{14}|^2 - |a_{23}|^2}{a_{11} + a_{22}}$$

4th row:

$$a_{41}\nu_1 - a_{13}\nu_3 + a_{42}\nu_5 = 0 \quad \Leftrightarrow \quad a_{22}a_{33}|a_{14}|^2 = a_{11}a_{44}|a_{23}|^2$$

6th row:

$$a_{33} - a_{22} = -a_{24}\nu_3 + a_{32}\nu_4 + a_{31}\nu_5 = 0 \quad \Leftrightarrow \quad (a_{33} - a_{22})\left(1 - \frac{|a_{23}|^2}{a_{22}a_{33}}\right) = \frac{|a_{13}|^2 - |a_{24}|^2}{a_{11} + a_{22}}$$

7th row:

$$-a_{23}\nu_3 + a_{42}\nu_4 + a_{41}\nu_5 = 0 \quad \Leftrightarrow \quad a_{22}a_{44}|a_{13}|^2 = a_{11}a_{33}|a_{24}|^2$$

Remark. The equations $-a_{11}a_{24}a_{32} = a_{22}a_{14}a_{31}$ and $-a_{33}a_{24}a_{41} = a_{44}a_{23}a_{31}$ obtained earlier have been used to simplify the compatibility equations for 4th and 7th rows. Then the latter equations are in turn used for simplifying the compatibility equations for 3rd and 6th rows. We also have used the following lemma.

Lemma 5.3.

$$(a_{33} - a_{11})(a_{11} + a_{22}) = a_{33}a_{22} - a_{11}a_{44}$$

and similar equations hold when switching the subscripts by $1 \leftrightarrow 2$ *or* $3 \leftrightarrow 4$.

Proof.

$$\begin{aligned}(a_{33} - a_{11})(a_{11} + a_{22}) &= a_{33}(a_{11} + a_{22}) - a_{11}(a_{11} + a_{22})\\ &= a_{33}(a_{11} + a_{22}) - a_{11}(a_{33} + a_{44})\\ &= a_{33}a_{22} - a_{11}a_{44}\end{aligned}$$

□

Now we gather all compatibility equations obtained earlier and keep the independent ones.

Complete set of compatibility equations

$$a_{11} + a_{22} = a_{33} + a_{44}$$

$$-a_{11}a_{24}a_{32} = a_{22}a_{14}a_{31}$$

$$-a_{33}a_{24}a_{41} = a_{44}a_{23}a_{31}$$

$$(a_{11} - a_{33})\left(1 - \frac{|a_{13}|^2}{a_{11}a_{33}}\right) = \frac{|a_{14}|^2 - |a_{23}|^2}{a_{11} + a_{22}}$$

$$(a_{33} - a_{22})\left(1 - \frac{|a_{23}|^2}{a_{22}a_{33}}\right) = \frac{|a_{13}|^2 - |a_{24}|^2}{a_{11} + a_{22}}$$

Since we are in **Case I**, we see from the 2nd equation that $a_{32} \neq 0$ and $a_{14} \neq 0$. Now the 2nd and 3rd equations give

$$\frac{|a_{13}|^2}{a_{11}a_{33}} = \frac{|a_{24}|^2}{a_{22}a_{44}} \qquad \text{and} \qquad \frac{|a_{23}|^2}{a_{22}a_{33}} = \frac{|a_{14}|^2}{a_{11}a_{44}}. \qquad (*)$$

Using these, together with Lemma 5.3, we can further simplify the last two equations and finally obtain

$$a_{11} + a_{22} = a_{33} + a_{44}$$

$$-a_{11}a_{24}a_{32} = a_{22}a_{14}a_{31}$$

$$-a_{33}a_{24}a_{41} = a_{44}a_{23}a_{31}$$

$$(a_{11} - a_{33})\left(1 - \frac{|a_{13}|^2}{a_{11}a_{33}} - \frac{|a_{23}|^2}{a_{22}a_{33}}\right) = 0$$

$$(a_{33} - a_{22})\left(1 - \frac{|a_{13}|^2}{a_{11}a_{33}} - \frac{|a_{23}|^2}{a_{22}a_{33}}\right) = 0$$

If

$$1 - \frac{|a_{13}|^2}{a_{11}a_{33}} - \frac{|a_{23}|^2}{a_{22}a_{33}} \neq 0,$$

then we have $a_{11} - a_{33} = a_{33} - a_{22} = 0$, which implies $a_{11} = a_{22} = a_{33} = a_{44}$ and $(*)$ gives $|a_{13}|^2 = |a_{24}|^2$ and $|a_{23}|^2 = |a_{14}|^2$. The remaining equations are

$$-a_{24}a_{32} = a_{14}a_{31} \qquad \text{and} \qquad -a_{24}a_{41} = a_{23}a_{31}.$$

Hence, if we write

$$a_{11} = a_{22} = a_{33} = a_{44} = a$$

$$a_{13} = \rho e^{i\theta_{13}}, \qquad a_{14} = \sigma e^{i\theta_{14}}, \qquad a_{23} = \sigma e^{i\theta_{23}}, \qquad a_{24} = \rho e^{i\theta_{24}},$$

where $a, \rho, \sigma \in \mathbb{R}^*$ and $\theta_{13}, \theta_{23}, \theta_{14}, \theta_{24} \in \mathbb{R}$, then we only need

$$\theta_{13} + \theta_{24} = \theta_{14} + \theta_{23} + \pi.$$

Now, if

$$1 - \frac{|a_{13}|^2}{a_{11}a_{33}} - \frac{|a_{23}|^2}{a_{22}a_{33}} = 0,$$

we let $\frac{|a_{13}|^2}{a_{11}a_{33}} = k$ and $\frac{|a_{23}|^2}{a_{22}a_{33}} = \ell$. Then from $(*)$ we also have $\frac{|a_{24}|^2}{a_{22}a_{44}} = k$ and $\frac{|a_{14}|^2}{a_{11}a_{44}} = \ell$. If we further let

$$a_{11} = a, \qquad a_{22} = b, \qquad a_{33} = c, \qquad a_{44} = d,$$

then $k, \ell, a, b, c, d \in \mathbb{R}$ need to satisfy

$$a + b = c + d \quad \text{and} \quad k + \ell = 1,$$

with $ack, ad\ell, bc\ell \in \mathbb{R}^+$.

Finally, if we write

$$a_{13} = \sqrt{ack}e^{i\theta_{13}}, \qquad a_{14} = \sqrt{ad\ell}e^{i\theta_{14}}, \qquad a_{23} = \sqrt{bc\ell}e^{i\theta_{23}}, \qquad a_{24} = \sqrt{bdk}e^{i\theta_{24}},$$

where $\theta_{13}, \theta_{14}, \theta_{23}, \theta_{24} \in \mathbb{R}$. Then, the compatibility equations are satisfied if

$$\theta_{13} + \theta_{24} = \theta_{14} + \theta_{23} + \pi.$$

Solutions for Case I.1

To summarize, we have

$$\begin{pmatrix} a_{11} & 0 & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & 0 & a_{44} \end{pmatrix} = \begin{pmatrix} a & 0 & \sqrt{ack}e^{i\theta_{13}} & \sqrt{ad\ell}e^{i\theta_{14}} \\ 0 & b & \sqrt{bc\ell}e^{i\theta_{23}} & \sqrt{bdk}e^{i\theta_{24}} \\ \sqrt{ack}e^{-i\theta_{13}} & \sqrt{bc\ell}e^{-i\theta_{23}} & c & 0 \\ \sqrt{ad\ell}e^{-i\theta_{14}} & \sqrt{bdk}e^{-i\theta_{24}} & 0 & d \end{pmatrix},$$

where $a, b, c, d, k, \ell, \theta_{13}, \theta_{14}, \theta_{23}, \theta_{24} \in \mathbb{R}$, such that

$$\begin{cases} ack, ad\ell, bc\ell \in \mathbb{R}^+ \\ a + b = c + d \\ k + \ell = 1 \\ \theta_{13} + \theta_{24} = \theta_{14} + \theta_{23} + \pi. \end{cases} \qquad \text{or} \qquad a = b = c = d$$

5.1.2 Sub-case I.2: $a_{11} + a_{22} = 0$

In this sub-case, the six columns chosen at the beginning of **Sub-case I.1** are no longer linearly independent. In fact, if they were linearly independent, then by going through the same linear algebra as in **First column** there, we immediately deduce that $\mathrm{rank}(B_A) \geq 7$.

On the other hand, by the argument for linear independence given at the beginning of **Sub-case I.1**, we see that we still have $\mathrm{rank}(\mathbf{b_1}, \mathbf{b_3}, \mathbf{b_5}, \mathbf{b_2}, \mathbf{b_{10}}) = 5$. In addition to these five, we now choose $\mathbf{b_4}$ (instead of $\mathbf{b_9}$), which is easily seen to be linearly independent of $\{\mathbf{b_1}, \mathbf{b_3}, \mathbf{b_5}, \mathbf{b_2}, \mathbf{b_{10}}\}$. Hence, we can now carry a similar process as in **Sub-case I.1** to get other equations on a_{ij} using the six linearly independent columns $\{\mathbf{b_1}, \mathbf{b_3}, \mathbf{b_4}, \mathbf{b_5}, \mathbf{b_2}, \mathbf{b_{10}}\}$.

The condition $\mathrm{rank}(B_A) = 6$ now implies $\{\mathbf{b_9}, \mathbf{b_7}, \mathbf{b_8}, \mathbf{b_6}\} \subset \mathrm{Span}\,\{\mathbf{b_1}, \mathbf{b_3}, \mathbf{b_4}, \mathbf{b_5}, \mathbf{b_2}, \mathbf{b_{10}}\}$.

Thus,

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a_{14} & 0 & a_{13} \\ -a_{14} & 0 & -a_{13} & 0 \\ -a_{13} & 0 & 0 & 0 \\ \\ 0 & a_{24} & 0 & a_{23} \\ -a_{24} & 0 & -a_{23} & a_{33}-a_{22} \\ -a_{23} & a_{44}-a_{22} & 0 & 0 \\ \\ 0 & 0 & -a_{33} & -a_{32} \\ -a_{33}-a_{44} & -a_{32} & 0 & -a_{42} \\ 0 & -a_{42} & 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{13} & a_{14} & 0 & 0 & 0 \\ 0 & a_{23} & a_{24} & 0 & 0 & 0 \\ a_{31} & a_{33}-a_{11} & 0 & 0 & a_{32} & 0 \\ a_{41} & 0 & a_{44}-a_{11} & 0 & a_{42} & -a_{14} \\ \\ 0 & 0 & 0 & a_{22} & 0 & 0 \\ 0 & 0 & 0 & a_{32} & a_{31} & 0 \\ 0 & 0 & 0 & a_{42} & a_{41} & -a_{24} \\ \\ 0 & -a_{31} & 0 & 0 & 0 & 0 \\ 0 & -a_{41} & -a_{31} & 0 & 0 & 0 \\ 0 & 0 & -a_{41} & 0 & 0 & -a_{44} \end{pmatrix} \mathbf{X},$$

for some unique matrix $\mathbf{X} = \begin{pmatrix} \gamma_1 & \delta_1 & \mu_1 & \nu_1 \\ \gamma_2 & \delta_2 & \mu_2 & \nu_2 \\ \gamma_3 & \delta_3 & \mu_3 & \nu_3 \\ \gamma_4 & \delta_4 & \mu_4 & \nu_4 \\ \gamma_5 & \delta_5 & \mu_5 & \nu_5 \\ \gamma_6 & \delta_6 & \mu_6 & \nu_6 \end{pmatrix} \in M(6,4;\mathbb{C})$.

First column

$$\begin{array}{lllllll} & \text{5th row gives} & \gamma_4 = 0 & \Longrightarrow & \text{8th row gives} & \gamma_2 = 0 \\ \Longrightarrow & \text{2nd row gives} & \gamma_3 = 0 & \Longrightarrow & \text{1st row gives} & \gamma_1 = 0 \\ \Longrightarrow & \text{6th row gives} & \gamma_5 = \dfrac{-a_{24}}{a_{31}} & \Longrightarrow & \text{7th row gives} & \gamma_6 = \dfrac{a_{23}a_{31} - a_{24}a_{41}}{a_{24}a_{31}} \end{array}$$

Compatibility equations

3rd row:

$-a_{14} = a_{32}\gamma_5 \quad \Leftrightarrow \quad a_{14}a_{31} = a_{24}a_{32}$

4th row:

$-a_{13} = a_{42}\gamma_5 - a_{14}\gamma_6 \quad \Leftrightarrow \quad |a_{13}|^2 + |a_{14}|^2 = |a_{23}|^2 + |a_{24}|^2$

9th row:

$-a_{33} - a_{44} = 0 \quad \Leftrightarrow \quad a_{33} + a_{44} = 0$

10th row:

$-a_{44}\gamma_6 = 0 \quad \Leftrightarrow \quad a_{44}(a_{23}a_{31} - a_{24}a_{41}) = 0$

Lemma 5.4. $a_{14}a_{23} \neq 0$

Proof. Since we are in **Case I**, the result follows easily from the first compatibility equation. □

Second column

$$
\begin{aligned}
&\text{8th row gives} \quad \delta_2 = 0 && \Longrightarrow \text{5th row gives} \quad \delta_4 = \frac{a_{24}}{a_{22}} \\
\Longrightarrow\ &\text{6th row gives} \quad \delta_5 = \frac{-a_{24}a_{32}}{a_{22}a_{31}} && \Longrightarrow \text{9th row gives} \quad \delta_3 = \frac{a_{32}}{a_{31}} \\
\Longrightarrow\ &\text{1st row gives} \quad \delta_1 = \frac{-a_{14}a_{32}}{a_{11}a_{31}} && \Longrightarrow \text{7th row gives}
\end{aligned}
$$

$$\delta_6 = \frac{1}{a_{24}} \left(a_{22} - a_{44} + \frac{|a_{24}|^2}{a_{22}} - \frac{a_{24}a_{32}a_{41}}{a_{22}a_{31}} \right)$$

Compatibility equations

(We will take into account $a_{14} \neq 0$ and $a_{23} \neq 0$, which are obtained earlier.)

2nd row:

$a_{14} = a_{24}\delta_3 \quad \Leftrightarrow \quad a_{14}a_{31} = a_{24}a_{32}$

3rd row:

$a_{31}\delta_1 + a_{32}\delta_5 = 0 \quad \Leftrightarrow \quad a_{14}a_{31} = a_{24}a_{32}$

4th row:

$a_{41}\delta_1 + (a_{44} - a_{11})\delta_3 + a_{42}\delta_5 - a_{14}\delta_6 = 0 \quad \Leftrightarrow \quad |a_{14}|^2 - |a_{24}|^2 - a_{11}a_{44} = 0$

10th row:

$-a_{42} = -a_{41}\delta_3 - a_{44}\delta_6 \quad \Leftrightarrow \quad (a_{22} - a_{44})\left(|a_{14}|^2 - |a_{24}|^2 - a_{11}a_{44}\right) = 0$

Third column

$$
\begin{aligned}
&\text{5th row gives} \quad \mu_4 = 0 && \Longrightarrow \text{6th row gives} \quad \mu_5 = \frac{-a_{23}}{a_{31}} \\
\Longrightarrow\ &\text{8th row gives} \quad \mu_2 = \frac{a_{33}}{a_{31}} && \Longrightarrow \text{9th row gives} \quad \mu_3 = \frac{-a_{33}a_{41}}{a_{31}^2} \\
\Longrightarrow\ &\text{6th row gives} \quad \mu_6 = \frac{-a_{23}a_{41}}{a_{24}a_{31}} && \Longrightarrow \text{1st row gives} \quad \mu_1 = \frac{a_{33}}{a_{11}a_{31}} \left(\frac{|a_{14}|^2}{a_{31}} - a_{13} \right)
\end{aligned}
$$

Compatibility equations

(We will take into account $a_{14} \neq 0$ and $a_{23} \neq 0$, which are obtained earlier.)

2nd row:

$$a_{23}\mu_2 + a_{24}\mu_3 = 0 \quad \Leftrightarrow \quad a_{44}(a_{23}a_{31} - a_{24}a_{41}) = 0$$

3rd row:

$$-a_{13} = a_{31}\mu_1 + (a_{33} - a_{11})\mu_2 + a_{32}\mu_5 \quad \Leftrightarrow \quad (a_{33} - a_{11})(|a_{13}|^2 - a_{11}a_{33}) = a_{33}|a_{14}|^2 - a_{11}|a_{23}|^2$$

4th row:

$$a_{41}\mu_1 + (a_{44} - a_{11})\mu_3 + a_{42}\mu_5 - a_{14}\mu_6 = 0 \quad \Leftrightarrow \quad (a_{44} - a_{11})(|a_{14}|^2 - a_{11}a_{44}) = a_{44}|a_{13}|^2 - a_{11}|a_{24}|^2$$

10th row:

$$-a_{41}\mu_3 - a_{44}\mu_6 = 0 \quad \Leftrightarrow \quad a_{44}(a_{23}a_{31} - a_{24}a_{41}) = 0$$

Fourth column

$$\text{5th row gives} \quad \nu_4 = \frac{a_{23}}{a_{22}} \quad \Longrightarrow \quad \text{8th row gives} \quad \nu_2 = \frac{a_{32}}{a_{31}}$$

$$\Longrightarrow \quad \text{9th row gives} \quad \nu_3 = \frac{1}{a_{31}}\left(a_{42} - \frac{a_{32}a_{41}}{a_{31}}\right)$$

$$\Longrightarrow \quad \text{6th row gives} \quad \nu_5 = \frac{1}{a_{31}}\left(a_{33} - a_{22} - \frac{|a_{23}|^2}{a_{22}}\right)$$

$$\Longrightarrow \quad \text{1st row gives} \quad \nu_1 = \frac{1}{a_{11}}\left[\frac{a_{14}}{a_{31}}\left(\frac{a_{32}a_{41}}{a_{31}} - a_{42}\right) - \frac{a_{13}a_{32}}{a_{31}}\right]$$

$$\Longrightarrow \quad \text{7th row gives} \quad \nu_6 = \frac{1}{a_{24}}\left[\frac{a_{41}}{a_{31}}\left(a_{33} - a_{22} - \frac{|a_{23}|^2}{a_{22}}\right) + \frac{a_{23}a_{42}}{a_{22}}\right]$$

Compatibility equations

(We will take into account $a_{14} \neq 0$ and $a_{23} \neq 0$, which are obtained earlier.)

2nd row:

$$a_{13} = a_{23}\nu_2 + a_{24}\nu_3 \quad \Leftrightarrow \quad a_{14}a_{31} = a_{24}a_{32}$$

3rd row:

$$a_{31}\nu_1 + (a_{33} - a_{11})\nu_2 + a_{32}\nu_5 = 0 \quad \Leftrightarrow \quad |a_{14}|^2 + |a_{23}|^2 - |a_{13}|^2 - |a_{24}|^2 = 2a_{11}a_{44}$$

4th row:

$$a_{41}\nu_1 + (a_{44} - a_{11})\nu_3 + a_{42}\nu_5 - a_{14}\nu_6 = 0 \quad \Leftrightarrow \quad (|a_{24}|^2 - |a_{14}|^2)\,(|a_{13}|^2 + |a_{23}|^2 - |a_{14}|^2) = 0$$

10th row:

$$-a_{41}\nu_3 - a_{44}\nu_6 = 0 \quad \Leftrightarrow \quad (a_{33} - a_{22})(|a_{24}|^2 - a_{22}a_{44}) = a_{33}|a_{23}|^2 - a_{22}|a_{14}|^2$$

Now we gather all compatibility equations obtained in this sub-case and keep the independent ones.

Complete set of compatibility equations

$$a_{33} + a_{44} = 0$$

$$a_{14}a_{31} = a_{24}a_{32}$$

$$|a_{13}|^2 + |a_{14}|^2 = |a_{23}|^2 + |a_{24}|^2$$

$$a_{44}(|a_{14}|^2 - |a_{23}|^2) = 0$$

$$|a_{14}|^2 - |a_{24}|^2 = a_{11}a_{44}$$

$$\left(|a_{14}|^2 - |a_{24}|^2\right)\left(|a_{13}|^2 + |a_{23}|^2 - |a_{14}|^2\right) = 0$$

Before solving the equations, we first prove that:

Lemma 5.5. $a_{33} = a_{44} = 0$

Proof. Suppose $a_{44} \neq 0$, then the fourth equation above implies that $|a_{14}| = |a_{23}|$. From the last equation we get $|a_{14}| = |a_{24}|$, which contradicts the fifth equation.Thus, $a_{44} = 0$ and hence $a_{33} = 0$ from the first equation. □

By Lemma 5.5, our equations now become

$$a_{33} = a_{44} = 0$$

$$a_{14}a_{31} = a_{24}a_{32}$$

$$|a_{13}|^2 + |a_{14}|^2 = |a_{23}|^2 + |a_{24}|^2$$

$$|a_{14}|^2 = |a_{24}|^2$$

Solutions for Case I.2

It is now easy to write down the solutions for this sub-case.

$$\begin{pmatrix} a & 0 & re^{i\theta_{13}} & re^{i\theta_{14}} \\ 0 & -a & re^{i\theta_{23}} & re^{i\theta_{24}} \\ re^{-i\theta_{13}} & re^{-i\theta_{23}} & 0 & 0 \\ re^{-i\theta_{14}} & re^{-i\theta_{24}} & 0 & 0 \end{pmatrix},$$

where $a, r \in \mathbb{R}^*$ and $\theta_{13}, \theta_{14}, \theta_{23}, \theta_{24} \in \mathbb{R}$ satisfying

$$\theta_{13} + \theta_{24} = \theta_{14} + \theta_{23}.$$

5.2 Case II

By exploiting the symmetries $(z_1, z_2, z_3, z_4) \mapsto (z_1, z_2, z_4, z_3)$ of $D_{2,2}$ and $(z_1, z_2, z_3, z_4) \mapsto (z_2, z_1, z_3, z_4)$ of $D_{2,2}$, we may divide it into three cases:

$$a_{13} = a_{23} = 0 \qquad \text{or} \qquad a_{13} = a_{14} = 0 \qquad \text{or} \qquad a_{14} = a_{23} = 0.$$

and other elements in each sub-case are non-zero.

5.2.1 Sub-case II.1: $a_{13} = a_{23} = 0$

Solutions for Case II.1

$$\begin{pmatrix} a & 0 & 0 & re^{i\theta} \\ 0 & b & 0 & \rho e^{i\phi} \\ 0 & 0 & 0 & 0 \\ re^{-i\theta} & \rho e^{-i\phi} & 0 & a+b \end{pmatrix},$$

where $a, b, r, \rho \in \mathbb{R}^*$ and $\theta, \phi \in \mathbb{R}$ satisfying

$$\frac{r^2}{a} + \frac{\rho^2}{b} = a + b.$$

5.2.2 Sub-case II.2: $a_{13} = a_{14} = 0$

There is no solution in this sub-case.

5.2.3 Sub-case II.3: $a_{14} = a_{23} = 0$

Solutions for Case II.3

There are two kinds of solutions. The first kind is

$$\begin{pmatrix} a & 0 & re^{i\theta} & 0 \\ 0 & b & 0 & re^{i\phi} \\ re^{-i\theta} & 0 & b & 0 \\ 0 & re^{-i\phi} & 0 & a \end{pmatrix},$$

where $\theta, \phi \in \mathbb{R}$ and $a, b, r \in \mathbb{R}^*$ satisfying $(a-b)(ab-r^2) = 0$.

The second kind is

$$\begin{pmatrix} a & 0 & re^{i\theta} & 0 \\ 0 & -a & 0 & re^{i\phi} \\ re^{-i\theta} & 0 & b & 0 \\ 0 & re^{-i\phi} & 0 & -b \end{pmatrix},$$

where $a, r \in \mathbb{R}^*$ and $b, \theta, \phi \in \mathbb{R}$ satisfying $(a-b)(ab-r^2) = 0$.

5.3 Case III

By exploiting the symmetries $(z_1, z_2, z_3, z_4) \mapsto (z_1, z_2, z_4, z_3)$ of $D_{2,2}$ and $(z_1, z_2, z_3, z_4) \mapsto (z_2, z_1, z_3, z_4)$ of $D_{2,2}$, we may assume that $a_{13} = a_{14} = a_{23} = 0$ and $a_{24} \neq 0$.

Solutions for Case III

There are two kinds of solutions.

The first kind is

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & \sqrt{b^2 - a^2}e^{i\theta} \\ 0 & 0 & a & 0 \\ 0 & \sqrt{b^2 - a^2}e^{-i\theta} & 0 & b \end{pmatrix},$$

where $a, b, \theta \in \mathbb{R}$ satisfying $|b| > |a| > 0$.

The second kind is

$$\begin{pmatrix} a-b & 0 & 0 & 0 \\ 0 & b & 0 & abe^{i\theta} \\ 0 & 0 & 0 & 0 \\ 0 & abe^{-i\theta} & 0 & a \end{pmatrix},$$

where $a, b, \theta \in \mathbb{R}$ satisfying $ab(a-b) \neq 0$.

5.4 Case IV

There are three solutions:

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \qquad \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & -a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & -a \end{pmatrix}, \qquad \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & -a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $a \in \mathbb{R}^*$.

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