Proper Holomorphic Mappings among Bounded Symmetric Domains and related Hartogs Domains

Sui-Chung Ng[1], Zhenhan Tu[2*] & Wanke Yin[2]

[1] Department of Mathematics, East China Normal University
Shanghai 200241, P.R. China

[2] School of Mathematics and Statistics, Wuhan University
Wuhan, Hubei 430072, P.R. China

Email: scng@math.ecnu.edu.cn (S.-C. Ng), zhhtu.math@whu.edu.cn (Z. Tu)
& wankeyin@whu.edu.cn (W. Yin)

Abstract. The classical Alexander's Theorem states that every proper holomorphic self-mapping of a complex unit ball of dimension at least 2 is an automorphism. Since then, the study of proper holomorphic mappings has become an important topic in Several Complex Variables. Bounded symmetric domains, which include the complex unit balls, are among the most important domains in complex Euclidean spaces, due to the fact that they possess a lot of symmetries and are the universal covering spaces of various important mathematical objects. Henkin and Novikov proved that the analogue of Alexander's Theorem is also true for irreducible bounded symmetric domains of higher rank. These rigidity results for proper holomorphic mappings among bounded symmetric domains have been, by the efforts of a lot of people, extended to the cases with positive co-dimension or rank difference. The purpose of this article is to give a survey for these developments. In addition, we also include a section discussing some generalizations to the Hartogs domains over irreducible bounded symmetric domains

Key words: Automorphisms, bounded symmetric domains, Hartogs domains, proper holomorphic mappings, rigidity

2010 Mathematics Subject Classification: Primary 32H35, 32M15

1 Introduction

A continuous mapping $F : X \to Y$ of topological spaces is called proper if $F^{-1}(K)$ is a compact subset of X whenever K is a compact subset of Y. If X and Y are complex spaces and $F : X \to Y$ is a proper holomorphic mapping, then $F^{-1}(y)$ is a compact subvariety of X for all points y in Y. Therefore, when the complex space X is Stein, $F^{-1}(y)$ is finite for all points y in Y. A special class of proper holomorphic mappings are the biholomorphic mappings.

Remmert's Proper Mapping Theorem [Re1, Re2] asserts that if F is a proper holomorphic mapping between the complex spaces X and Y, then the image $B := F(A)$ is an analytic subset of Y for any analytic subset A of X. Moreover, when X is Stein, there exists a nowhere dense analytic subset $E \subset B$ such that $B \setminus E$ and $A \setminus F^{-1}(E)$ are complex manifolds and the restriction $F : A \setminus F^{-1}(E) \to B \setminus E$

*Corresponding author.

is a finitely sheeted holomorphic covering projection. Therefore, the existence of a non-trivial proper map from one complex space X to another Y often places strong restrictions on these spaces. Furthermore, even when proper maps exist, they tend to be rigid and rather few in number. For instance, if $n \geq 2$, a proper holomorphic map of the unit ball in $\mathbb{C}^n$ into itself is an analytic automorphism. Therefore, the study of proper holomorphic maps between complex spaces is of great importance.

The subject of bounded symmetric domains plays an important role in complex differential geometry. Various rigidity phenomena have been obtained for their quotient spaces. For instance, the strong rigidity of Siu [Siu1, Siu2], the Hermitian metric rigidity of Mok [Mok1, Mok2] and the super rigidity hence obtained (c.f. [Mok]). In this article, we will give a survey for the rigidity of proper holomorphic mappings among bounded symmetric domains, starting from the classical Alexander's Theorem for complex unit balls of dimension at least two.

2 Proper holomorphic mappings between unit balls

In this section, we will focus on the rigidity of proper holomorphic maps between bounded symmetric domains of rank one, i.e. the complex unit balls.

Write $\mathbb{B}^n$ for the unit ball in the complex space $\mathbb{C}^n$. The study of proper holomorphic maps between balls in complex Euclidean spaces can date back to the pioneering work of Poincaré [Po], who discovered that any biholomorphic map between two connected open pieces of the unit sphere in $\mathbb{C}^2$ is the restriction of a certain automorphism of $\mathbb{B}^2$. In his famous paper [Alx], Alexander proved that Poincaré's result also holds for complex unit balls in higher dimensions.

Theorem 2.1 (Alexander) *Any proper holomorphic self-map of $\mathbb{B}^n$ with $n > 1$ is an automorphism.*

Webster, in [W], first considered the geometric structure of proper holomorphic maps between balls in complex spaces of different dimensions. By the combining efforts of Cima-Suffridge [CS1] , Faran [Fa2] and Forstneric [Fo1] etc., we know that any proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $N < 2n - 1$, that is C^{N-n+1}-regular up to the boundary, is a totally geodesic embedding with respect to the Bergman metrics. Recall that two proper holomorphic maps f, g from $\mathbb{B}^n$ into $\mathbb{B}^N$ are said to be equivalent if there are $\sigma \in \mathrm{Aut}(\mathbb{B}^n)$ and $\tau \in \mathrm{Aut}(\mathbb{B}^N)$ such that $g = \tau \circ f \circ \sigma$. A proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^N$ is said to be linear or totally geodesic if it is equivalent to the standard big circle embedding $L(z) : z \to (z, 0)$. It is conjectured that the rigidity property still holds true for maps C^2-regular up to the boundary. By applying a very different method from the previous works, Huang [Hu1] gave an affirmative answer to this conjecture:

Theorem 2.2 (Huang) *Any proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $N < 2n - 1$, that is C^2-regular up to the boundary, is a totally geodesic embedding.*

Write $\mathcal{I}_1 = [n+1, 2n-2]$. The theorem of Huang says that there is no new proper holomorphic map added when the target dimension $N \in \mathcal{I}_1$. We call $\mathcal{I}_1$ the first gap interval for proper holomorphic mappings between balls. We mention that the discovery of inner functions can be used to show that there is a proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^{n+1}$, which can not be C^2-smooth at any boundary point. (See [HS], [Low], [Fo2], [Ste], etc).

The structure of proper holomorphic maps gets more complicated when $N \geq 2n-1$. In [Fa1], Faran classified proper holomorphic maps from $\mathbb{B}^2$ into $\mathbb{B}^3$, which are C^3-smooth up to the boundary. He obtained the following four different inequivalent proper holomorphic maps

$$(z, w, 0),\ (z, zw, w^2),\ (z^2, \sqrt{zw}, w^2),\ (z^3, \sqrt[3]{zw}, w^3).$$

However, the only embeddings are linear maps.

In [HJ], Huang-Ji showed that any proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^{2n-1}$ with $n \geq 3$, which is C^2-smooth up to the boundary, is either linear or equivalent to the Whitney map

$$W : z = (z_1, \cdots, z_n) = (z', z_n) \to (z_1, \cdots, z_{n-1}, z_n z) = (z', z_n z). \tag{1}$$

Since the Whitney map is not an immersion, together with the aforementioned work of Faran [Fa1], this shows that any proper holomorphic *embedding* from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $N = 2n-1$, which is twice continuously differentiable up to the boundary, must be a linear map. Earlier, D'Angelo constructed the following family F_θ of mutually inequivalent proper quadratic monomial maps from $\mathbb{B}^n$ into $\mathbb{B}^{2n}$ (See [DA]):

$$F_\theta(z', z_n) = (z', (\cos\theta) z_n, (\sin\theta) z_1 z_n, \cdots, (\sin\theta) z_{n-1} z_n, (\sin\theta) z_n^2),\ 0 < \theta \leq \pi/2. \tag{2}$$

Notice that by adding $N - 2n$ zero components to the D'Angelo map F_θ, we get a proper monomial embedding from $\mathbb{B}^n$ into $\mathbb{B}^N$ for any $N \geq 2n$. The combining effort in [Fa2] and [HJ] gives a complete description to the linearity problem for proper holomorphic embeddings from $\mathbb{B}^n$ into $\mathbb{B}^N$, which are C^2-smooth up to the boundary. However, in applications, one still hopes to get the linearity for mappings with a rich geometric structure. For instance, the following difficult problem initiated from the work of Siu [Siu3] and others is still open: (See Cao-Mok [CMk] for the work when $N \leq 2n-1$.)

Conjecture 2.3 : *Let f be a proper holomorphic mapping from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $1 < n < N$. Write $M = f(\mathbb{B}^n)$. Suppose that there is a subgroup Γ of $Aut(\mathbb{B}^N)$ such that (1) for any $\sigma \in \Gamma$, $\sigma(M) = M$; (2) M/Γ is compact. Then f is a linear embedding.*

In [Hu2], Huang obtained a complete description of the partial linearity for proper holomorphic maps between balls for $N < \frac{n(n+1)}{2}$, based on a very subtle analysis of the moving point trick and the larger symmetry property of the ball, as first used in [Hu1]. This result has found many immediate applications, which, in particular, shows that any proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $N \leq n(n+1)/2$, that is C^3-smooth up to the boundary, must be rational ([HJX2]).

In a recent paper of Hamada [Ha1], based on a careful analysis on the Chern-Moser normal form method (see [CM]) as developed in [Hu1] and [HJ], Hamada classified all proper rational maps from $\mathbb{B}^n$ into $\mathbb{B}^{2n}$ with $n \geq 4$. After the work of Hamada, Huang-Ji-Xu in [HJX1] proved that

Theorem 2.4 (Huang-Ji-Xu) *Any proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $4 \le n \le N \le 3n-4$, that is C^3-smooth up to the boundary, is equivalent to either the map $(W, 0')$ or $(F_\theta, 0')$ with $\theta \in [0, \pi/2)$.*

An immediate consequence of the work in [HJX1] is that there is no new map added when $N \in \mathcal{I}_2$ with $\mathcal{I}_2 := [2n+1, 3n-4]$. Since there are proper monomial maps from $\mathbb{B}^n$ into $\mathbb{B}^N$ for $3n-3 \le N \le 3n$ or $2n-1 \le N \le 2n$, that are not equivalent to maps of the form $(G, 0')$, we call $\mathcal{I}_2$ the second gap interval for proper holomorphic maps between balls.

By [HJY1], for any N with $3n-3 \le N \le 3n$ or $4n-6 \le N \le 4n$, there are many proper monomial maps from $\mathbb{B}^n$ into $\mathbb{B}^N$, that are not equivalent to maps of the form $(G, 0')$. Subsequently, Huang-Ji-Yin in [HJY] provides a third gap interval $\mathcal{I}_3 := [3n+1, 4n-7]$ for proper holomorphic maps between balls:

Theorem 2.5 (Huang-Ji-Yin) *Let F be a proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $n > 7$ and $3n+1 \le N \le 4n-7$. Assume that F is C^3-regular up to the boundary. Then F is equivalent to a map of the form $(G, 0')$, where G is a proper rational map from $\mathbb{B}^n$ into $\mathbb{B}^{3n}$.*

More generally, for any $n \ge 3$, write $K(n)$ for the largest positive integer m such that $m(m+1)/2 < n$. For each $1 \le k \le K(n)$, define $\mathcal{I}_k := [kn+1, (k+1)n - \frac{k(k+1)}{2} - 1]$. Then $\mathcal{I}_k$ is a closed interval containing positive integers if $n \ge 2 + \frac{k(k+1)}{2}$. Apparently, $\mathcal{I}_k \cap \mathcal{I}_{k'} = \emptyset$ for $k \ne k'$; and $\mathcal{I}_k$ for $k = 1, 2, 3$ are exactly the same intervals defined above. Write $\mathcal{I} = \cup_{k=1}^{K(n)} \mathcal{I}_k$. Then, for

$$\max_{N \in \mathcal{I}} N = (K(n)+1)n - \frac{K(n)(K(n)+1)}{2} - 1 \approx \frac{-1+\sqrt{1+8n}}{2} n - n - 1 \approx \sqrt{2} n^{\frac{3}{2}} - n - 1.$$

For any $N \notin \mathcal{I}$ (which certainly is the case when $N \ge 1.42 n^{\frac{3}{2}}$), by not a complicated construction, the authors obtained in [HJY1] many monomial proper holomorphic maps from $\mathbb{B}^n$ into $\mathbb{B}^N$, that can not be equivalent to maps of the form $(G, 0')$. Earlier in [DL], for $N \ge n^2 - 2n + 2$, D'Angelo and Lebl, by a different method, constructed a proper monomial map from $\mathbb{B}^n$ into $\mathbb{B}^N$, that is not equivalent to a map of the form $(G, 0')$. However, we have not been able to find a map, not equivalent to a map of the form $(G, 0')$, for $N \in \mathcal{I}$. Indeed, the first, the second and the third gap intervals mentioned above suggest the following conjecture:

Conjecture 2.6 (Huang-Ji-Yin [HJY]) *Let $n \ge 3$ be a positive integer, and let $\mathcal{I}_k$ $(1 \le k \le K(n))$ be defined above. Then any proper holomorphic rational map F from $\mathbb{B}^n$ into $\mathbb{B}^N$ is equivalent to a map of the form $(G, 0')$ if and only if $N \in \mathcal{I}_k$ for some $1 \le k \le K(n)$.*

As mentioned above, the necessary part follows from Theorem 2.8 of [HJY]; also the conjecture holds for $k = 1, 2, 3$. An affirmative solution to this gap conjecture would tells exactly for what pair (n, N) there are no new proper rational maps added.

Next, we describe briefly the idea for the proof of Theorem 2.5, and hope that it may motivate the general study of Conjecture 2.6. The proofs for the first and the second gaps are immediate applications of the much more precise classification results. When $N \in \mathcal{I}_3$, making a precise classification for all maps seems to be hard, the only known case is the classification of proper holomorphic maps from $\mathbb{B}^n$ to $\mathbb{B}^{3n-3}$ (See [AHJY]).

Consider the setting in the Heisenberg hypersurface case. Let F be a holomorphic map defined near 0 with $F(0) = 0$ into $\mathbb{C}^N$. Then the Taylor formula says that $F(z) = \sum_\alpha \frac{D^\alpha F}{\alpha!}(0) z^\alpha$. Hence the image of F stays in the linear subspace spanned by $\{D^\alpha F(0)\}_\alpha$. If $spann\{D^\alpha F(0)\}_\alpha \neq \mathbb{C}^N$, we get a gap from F. The crucial point in our argument is to find, for our map, a basis of $spann\{D^\alpha F(0)\}_\alpha$. The way to achieve is to get a good normal form for F. However, this is a highly non-linear normalization problem, for the maps need to satisfy the fundamental non-linear equation. While it is easy to get linear independent set from the first and the second jets (even for the most general case), finding more linearly independent elements to form a basis from the higher order jets is very involved. The basic tool at our disposal for this approach is the following lemma achieved by Huang [Hu1]:

Lemma 2.7 (Huang) *Let $\{\varphi_j\}_{j=1}^k$ and $\{\psi_j\}_{j=1}^k$ be holomorphic functions in $z \in \mathbb{C}^n$ near the origin. Assume that $\varphi_j(0) = \psi_j(0) = 0$ and $k < n$. Let $H(z, \overline{z})$ be a real analytic function for z near 0 such that*

$$\sum_{j=1}^{k} \varphi_j(z)\overline{\psi_j(z)} = |z|^2 H(z, \overline{z}). \tag{3}$$

Then $H(z, \overline{z}) \equiv 0$.

While analyzing the basic Chern-Moser equation, one finds that the assumption $N \in \mathcal{I}_3$ is exactly what we need, in several induction steps, for applying Lemma 2.7. For higher gap interval case, the equations derived from the Chern-Moser equation take the form (3) with $k \geq n$, and Lemma 2.7 is no longer applicable. One possible way to solve the general case is to transform these equations to the settings in Lemma 2.7, and use this lemma to determine all linearly independent elements, which can form a basis of $spann\{D^\alpha F(0)\}_\alpha$. See also [Eb] for a quite different method handling the gap phenomenon.

For other related problems, we refer the reader to [AHJY, HY1, HY2, HY3, HY4, Yin1] and references therein.

3 Proper holomorphic mappings among bounded symmetric domains of rank ≥ 2

Due to the big difference in boundary structures, the study of proper holomorphic mappings among bounded symmetric domains of rank at least 2 differs vastly from that of the complex unit balls. Let Ω be a bounded symmetric domain of rank at least 2 and rank$(\Omega) = r$. On one hand, the boundary $\partial\Omega$ of Ω is not smooth and consists of a number of strata. This causes the usual Cauchy-Riemann geometric methods on the unit sphere inapplicable. On the other hand, the smooth part (i.e. the stratum of the highest dimension) of $\partial\Omega$ is foliated by complex submanifolds that are themselves isomorphic to some other bounded symmetric domains of rank $r - 1$ (Readers can see Wolf [Wo] for the fine structure theory on the boundary of bounded symmetric domains). These boundary complex submanifolds and their counterparts in the interior of Ω turn out to be an important source of rigidity for proper holomorphic mappings.

The first result related to the rigidity of proper holomorphic maps among bounded symmetric domains of rank at least 2 is the following theorem given by Henkin-Tumanov [HT].

Theorem 3.1 (Henkin-Tumanov) *Let $\Omega \Subset \mathbb{C}^n$ be an irreducible bounded symmetric domain of rank at least 2. Denote by $Sh(\Omega) \subset \partial\Omega$ the Shilov boundary of Ω. Suppose $b \in Sh(\Omega)$. Let $U_b \subset \mathbb{C}^n$ be a connected open neighborhood of b in $\mathbb{C}^n$, and $f : U_b \to \mathbb{C}^n$ be an open holomorphic embedding such that $f(U_b \cap \Omega) = f(U_b) \cap \Omega$ and $f(U_b \cap Sh(\Omega)) = f(U_b) \cap Sh(\Omega)$. Then, there exists an automorphism $F : \Omega \to \Omega$ such that $F|_{U_b\cap\Omega} \equiv f|_{U_b\cap\Omega}$.*

The statement of Henkin-Tumanov by itself, *a priori*, does not imply that every proper holomorphic self-map of Ω must be an automorphism. If the map is assumed to extend locally across a point on the Shilov boundary, then one can show that such a map will respect the Shilov boundary and hence is an automorphism. This argument can be found in Mok-Ng [MN]. In the same article, the following theorem has also been obtained using the geometric structure defined by minimal rational curves on the compact dual of Ω. (For the general theory developed by Hwang-Mok for this geometric structure, one may consult the review articles [Hw, Mok3] and the references therein.)

Theorem 3.2 (Mok-Ng) *Let $\Omega \Subset \mathbb{C}^n$ be an irreducible bounded symmetric domain of rank at least 2. Suppose b is a smooth point on $\partial\Omega$. Let $U_b \subset \mathbb{C}^n$ be a connected open neighborhood of b in $\mathbb{C}^n$ and $f : U_b \to \mathbb{C}^n$ be an open holomorphic embedding such that $f(U_b \cap \Omega) \subset \Omega$ and $f(U_b \cap \partial\Omega) \subset \partial\Omega$. Then, there exists an automorphism $F : \Omega \to \Omega$ such that $F|_{U_b\cap\Omega} \equiv f|_{U_b\cap\Omega}$.*

For global proper holomorphic maps without assuming any boundary regularity, Henkin and Novikov [HN] obtained the following analogue of Alexander's Theorem.

Theorem 3.3 (Henkin-Novikov) *Let Ω be an irreducible bounded symmetric domain of rank at least 2. Then every proper holomorphic self-map of Ω is an automorphism.*

The next step after proper holomorphic self-maps is the equi-dimensional case, which is obtained by Tu [Tu].

Theorem 3.4 (Tu) *Let Ω, Ω' be irreducible bounded symmetric domains of rank at least 2 and of the same dimension. Then every proper holomorphic map from Ω to Ω' is a biholomorphism.*

Furthermore, Mok, Ng and Tu [MNT] obtained the following rigidity result concerning proper holomorphic mappings from bounded symmetric domains onto bounded convex domains.

Theorem 3.5 (Mok-Ng-Tu) *Let Ω be an irreducible bounded symmetric domain of rank ≥ 2 which is not isomorphic to a Type-IV classical symmetric domain of dimension at least 3. Let $F : \Omega \to D$ be a proper holomorphic map onto a bounded convex domain D. Then, $F : \Omega \to D$ is a biholomorphism and D is, up to an affine-linear transformation, the Harish-Chandra realization of Ω.*

3.1 Equal-rank case

In contrast to the complex unit balls, it happens that in the study of proper holomorphic maps among bounded symmetric domains of rank at least 2, the difficulty is not very much due to the codimension but the rank difference. Motivated by problems for locally Hermitian symmetric spaces of non-compact type, Mok conjectured that when Ω, Ω' are irreducible bounded symmetric domains with rank(Ω)=rank(Ω') ≥ 2, every proper holomorphic map from Ω to Ω' must be standard, i.e. totally geodesic with respect to the Bergman metrics. The conjecture has been proved by Tsai [Tsai].

Theorem 3.6 (Tsai) *Let $f : \Omega \to \Omega'$ be a proper holomorphic map, where Ω, Ω' are irreducible bounded symmetric domains and rank$(\Omega) = r$, rank$(\Omega') = r'$ are at least 2. If $r \geq r'$, then f is standard and $r = r'$.*

Thus, the above statement also implies that f exists only if $r \leq r'$.

The first important step towards to the proof of Tsai's Theorem is to show that f preserves a certain class of geodesic subspaces, called *maximal characteristic symmetric subspaces*. Although not mentioned explicitly, such property of f has already been proved in the earlier work of Mok-Tsai [MT]. The maximal characteristic symmetric subspaces are isomorphic to the boundary complex submanifolds mentioned at the beginning of this section and the latter could be regarded as the limits when the charateristic symmetric subspaces are pushed towards the boundary using automorphisms. Roughly speaking, the properness of f implies that f (after taking certain radial limits) preserves such boundary complex submanifolds and by using Cauchy integral-type arguments, one concludes that the maximal characteristic symmetric subspaces are also preserved by f.

Maximal characteristic symmetric subspaces of a bounded symmetric domain Ω in fact belong to a more general class of geodesic subspaces of Ω, called *invariantly geodesic subspaces*, whose definition is as follows. From the general theory of Hermitian symmetric spaces, we know that there exists a compact Hermitian symmetric space M (also called the compact dual of Ω) such that Ω can be embedded as an open submanifold of M. A subspace $D \subset \Omega$ is said to be invariantly geodesic if for every $g \in \mathrm{Aut}(M)$, we have $g(D) \cap \Omega$ totally geodesic in Ω (with respect to the Bergman metric of Ω) whenever the intersection is non-empty. In next section we will give a more detailed discussion on these subspaces specialized to Type-I domains. The original proof for Tsai's Theorem involves the classification for invariantly geodesic subspaces for all bounded symmetric domains. Very recently, Ng [Ng] has bypassed this classification and obtained a much shorter proof for Tsai's theorem.

3.2 Unequal-rank case for Type-I domains

By Tsai's Theorem, if $f : \Omega \to \Omega'$ is a proper holomorphic map, Ω being irreducible, and $\mathrm{rank}(\Omega)$, $\mathrm{rank}(\Omega') \geq 2$, then f can be non-standard only when $\mathrm{rank}(\Omega') > \mathrm{rank}(\Omega)$. Indeed, one can construct simple non-standard examples by just considering Type-I bounded symmetric domains. Here we recall that there are altogether four infinite families of irreducible bounded symmetric domains and two exceptional domains. Among them, Type-I domains are parametrized by two positive integers p, q, and are defined as $\Omega_{p,q} = \{Z \in M(p, q; \mathbb{C}) : I - Z\bar{Z}^t > 0\}$, where $M(p, q; \mathbb{C})$ is the set of p-by-q complex matrices and "> 0" stands for the positive-definiteness for Hermitian matrices. Thus, Type-I domains are generalizations of the complex unit balls. In what follows, we will restrict ourselves to Type-I domains as they already furnish a good platform for investigating many important properties of the proper holomorphic maps among bounded symmetric domains of higher rank.

Let $Z \in \Omega_{p,q}$. Then the map $Z \mapsto \begin{bmatrix} Z & 0 \\ 0 & g(Z) \end{bmatrix}$ gives a simple example of non-standard maps from $\Omega_{p,q}$ to $\Omega_{p',q'}$, where $p < p'$, $q < q'$ and $g : \Omega_{p,q} \to \Omega_{p'-p,q'-q}$ is a generic non-constant holomorphic map. We will in what follows call such maps and those equivalent (by composing with automorphisms) to them *diagonal-type*. Starting from this simple example, the following important question is naturally raised: (*) *Are all proper holomorphic maps among Type-I domains diagonal-type?* In particular, does there exist any proper holomorphic map if $p \leq p'$ and $q \leq q'$ are not satisfied? For the latter question, Tu [Tu1] and Mok [Mok4] have proved the non-existence for certain special cases and the general case is still open. Back to Question (*), there had been quite a long time that the general belief was towards the affirmative side and evidence was provided by the works of Tu [Tu1], Ng [Ng1] and Kim-Zaitsev [KZ], which include the following results.

Theorem 3.7 (Tu for $p + 1 = q = p'$, Ng for other cases) *Let $q \geq p \geq 2$. If $p' \leq \min(2p-1, q)$, then every proper holomorphic map $f : \Omega_{p,q} \to \Omega_{p',q}$ is standard.*

Theorem 3.8 (Kim-Zaitsev) *Let $q \geq p \geq 2$ and $f : \Omega_{p,q} \to \Omega_{p',q'}$ be a proper holomorphic map which extends smoothly to a neighborhood of a smooth boundary point of $\Omega_{p,q}$. Assume that $q' < 2q - 1$ and $p' < q$. Then $p' \geq p$, $q' \geq q$, and f is diagonal-type.*

On the other hand, Ng very recently has found examples that are not diagonal-type and Seo [Seo] has generalized these examples based on the construction by Ng. From these examples, it is now evident that even for Type-I domains the moduli space of proper holomorphic maps is far from being non-trivial and thus one would aim for some classifications. Seo [Seo] has made a preliminary trial on the classification for some very special cases. We mention at this point that the discovery of these examples and the works of Ng [Ng1], Seo [Seo] are based on the investigation of certain geodesic subspaces of Type-I domains. The rest of this section will be devoted to this approach to the study of proper holomorphic maps.

3.2.1 Maximal invariantly geodesic subspace and generalized ball

As mentioned earlier, maximal characteristic symmetric subspaces play an important role in the study of proper holomorphic maps. For a Type-I domain $\Omega_{p,q}$, they are precisely the subspaces equivalent under automorphisms to the subspace consisting of all $Z \in \Omega_{p,q}$ whose entries on the last row and the last column are zero. Thus, the maximal characteristic symmetric subspaces of $\Omega_{p,q}$ are isomorphic to $\Omega_{p-1,q-1}$. Now for every pair of positive integers (r, s) such that $r \le p$ and $s \le q$, one can similarly consider the subspaces equivalent under automorphisms to the subspace consisting of all $Z \in \Omega_{p,q}$ whose entries on the last $(p - r)$ rows and last $(q - s)$ columns are zero. These subspaces, which are isomorphic to $\Omega_{r,s}$, are precisely the aforementioned invariantly geodesic subspaces of $\Omega_{p,q}$. (See [Tsai].)

In [Ng1], it has been discovered that maximal invariantly geodesic subspaces are also important for the study of proper holomorphic maps. We first of all note that maps of diagonal-type preserve not only maximal characteristic symmetric subspaces but also maximal invariantly geodesic subspaces. In addition, it has been shown in [Ng1] that for $\Omega_{p,q}$, the moduli space of the maximal invariantly geodesic subspaces that are isomorphic to $\Omega_{p-1,q}$ (resp. $\Omega_{p,q-1}$) is the generalized ball $D_{p,q}$ (resp. $D_{q,p}$). Here we recall that the generalized ball is defined as

$$D_{p,q} = \left\{ [z_1, \ldots, z_{p+q}] \in \mathbb{P}^{p+q-1} : \sum_{i=1}^{p} |z_i|^2 > \sum_{j=p+1}^{p+q} |z_j|^2 \right\}.$$

In turn, $\Omega_{p,q}$ parametrizes the maximal projective linear subspaces in $D_{p,q}$. The relation to proper holomorphic maps of these statements is described in the following proposition, which is also the basis for the works in [Ng1, Seo].

Proposition 3.9 *Let $f : \Omega_{p,q} \to \Omega_{p',q'}$ be a proper holomorphic map. If the image of a general maximal invariantly geodesic subspace under f is contained in a unique maximal invariantly geodesic subspace in the target domain, then f induces a rational proper map $\tilde{f} : D_{p,q} \to D_{p',q'}$. Conversely, if $g : D_{p,q} \to D_{p',q'}$ is a rational proper map such that the image of a general maximal linear subspace is contained in a unique maximal linear subspace, then g also induces a proper holomorphic map $\tilde{g} : \Omega_{p,q} \to \Omega_{p',q'}$.*

We here remark that proper holomorphic maps among generalized balls must preserve the maximal linear subspaces inside the generalized balls [Ng2]. Therefore we see from this proposition that the proper holomorphic maps among Type-I domains are deeply related to those among generalized balls. As generalized balls have much simpler boundary structures, their proper holomorphic maps are a lot easier to construct and investigate. (See [BH, BEH, Ng2, Ng3] for related works.) Although at this point the exact correspondence between these two sets of proper maps are not completely known, the classification of proper maps among generalized balls will certainly give a lot of information regarding those among Type-I domains. Along this direction, Gao and Ng [GN] has recently classified all rational proper holomorphic maps of degree 2 from $D_{2,2}$ to $D_{3,3}$ and thereby found a new type of examples for Type-I domains that were completely out of reach by previous methods. Finally, we state the following conjecture, which is true for all known examples so far and its confirmation should be a ground-breaking point for the study for proper holomorphic maps among irreducible bounded symmetric domains of higher rank. The conjecture has been confirmed in some cases by Ng [Ng1].

Conjecture 3.10 *Let Ω, Ω' be irreducible bounded symmetric domains of rank at least 2 and $f : \Omega \to \Omega'$ be a proper holomorphic map. Then f maps maximal invariantly geodesic subspaces of Ω into maximal invariantly geodesic subspaces of Ω'.*

4 Proper holomorphic mappings between Hartogs domains over bounded symmetric domain

Hua domains, named after Chinese mathematician Loo-Keng Hua, are domains in $\mathbb{C}^n$ fibered over irreducible bounded symmetric domains with fibers being generalized complex ellipsoids. Hua domains are bounded pseudoconvex domains and are in general nonhomogeneous and without smooth boundaries. This section is devoted to the rigidity results on proper holomorphic mappings between two equidimensional Hua domains.

A generalized complex ellipsoid (also called generalized pseudoellipsoid) is a domain of the form

$$\Sigma(\mathbf{n};\mathbf{p}) = \left\{ (\zeta_1, \cdots, \zeta_r) \in \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_r} : \sum_{k=1}^{r} \|\zeta_k\|^{2p_k} < 1 \right\},$$

where $\mathbf{n} = (n_1, \cdots, n_r) \in \mathbb{N}^r$, $\mathbf{p} = (p_1, \cdots, p_r) \in (\mathbb{R}_+)^r$, and $\|\cdot\|$ is the standard Hermitian norm.

In the special case where all the $p_k = 1$, the generalized complex ellipsoid $\Sigma(\mathbf{n};\mathbf{p})$ reduces to the unit ball in $\mathbb{C}^{n_1+\cdots+n_r}$. Also, it is known that a generalized complex ellipsoid $\Sigma(\mathbf{n};\mathbf{p})$ is homogeneous if and only if $p_k = 1$ for all $1 \le k \le r$. In general, a generalized complex ellipsoid is not strongly pseudoconvex and its boundary is not smooth. By relabelling the coordinates, we can always assume that $p_2 \neq 1, \cdots, p_r \neq 1$, that is, there is at most one 1 in $p_1, \cdots, p_r$.

For the biholomorphic mappings between two equidimensional generalized complex ellipsoids, in 1968, Naruki [Nar] proved the following result.

Theorem 4.1 (Naruki) *Let $\Sigma(\mathbf{n};\mathbf{p})$ and $\Sigma(\mathbf{m};\mathbf{q})$ be two equidimensional generalized complex ellipsoids with $\mathbf{n}$, $\mathbf{m} \in \mathbb{N}^r$ and $\mathbf{p}$, $\mathbf{q} \in (\mathbb{R}_+)^r$ (where $p_k \neq 1$, $q_k \neq 1$ for $2 \le k \le r$). Then $\Sigma(\mathbf{n};\mathbf{p})$ is biholomorphic to $\Sigma(\mathbf{m};\mathbf{q})$ if and only if there exists a permutation $\sigma \in S_r$ (where S_r is the permutation group of the r numbers $\{1, \cdots, r\}$) such that $n_{\sigma(j)} = m_j$, $p_{\sigma(j)} = q_j$ for $1 \le j \le r$.*

The holomorphic automorphism group $\mathrm{Aut}(\Sigma(\mathbf{n};\mathbf{p}))$ of $\Sigma(\mathbf{n};\mathbf{p})$ has been studied by Dini-Primicerio [DP2], Kodama [Kod] and Kodama-Krantz-Ma [KKM]. In 2013, Kodama [Kod] obtained the following result.

Theorem 4.2 (Kodama) *(i) If 1 does not appear in $p_1, \cdots, p_r$, then any automorphism $\varphi \in \mathrm{Aut}(\Sigma(\mathbf{n};\mathbf{p}))$ is of the form*

$$\varphi(\zeta_1, \cdots, \zeta_r) = \left(\gamma_1(\zeta_{\sigma(1)}), \cdots, \gamma_r(\zeta_{\sigma(r)})\right), \tag{4}$$

where $\sigma \in S_r$ *is a permutation of the* r *numbers* $\{1, \cdots, r\}$ *such that* $n_{\sigma(i)} = n_i, p_{\sigma(i)} = p_i$ $(1 \le i \le r)$ *and* $\gamma_1, \cdots, \gamma_r$ *are unitary transformations of* $\mathbb{C}^{n_1}(n_{\sigma(1)} = n_1), \cdots, \mathbb{C}^{n_r}(n_{\sigma(r)} = n_r)$ *respectively.*

(ii) *If* 1 *appears in* $p_1, \cdots, p_r$*, we can assume, without loss of generality, that* $p_1 = 1, p_2 \neq 1, \cdots, p_r \neq 1$*, then* $\mathrm{Aut}(\Sigma(\mathbf{n}; \mathbf{p}))$ *is generated by elements of the form* (4) *and automorphisms of the form*

$$\varphi_a(\zeta_1, \zeta_2, \cdots, \zeta_r) = \left(T_a(\zeta_1), \zeta_2(\psi_a(\zeta_1))^{\frac{1}{2p_2}}, \cdots, \zeta_r(\psi_a(\zeta_1))^{\frac{1}{2p_r}}\right),$$

where T_a *is an automorphism of the ball* $\mathbf{B}^{n_1}$ *in* $\mathbb{C}^{n_1}$*, which brings a point* $a \in \mathbf{B}^{n_1}$ *in the origin and*

$$\psi_a(\zeta_1) = \frac{1 - \|a\|^2}{(1 - \langle \zeta_1, a \rangle)^2}.$$

Let Ω be an irreducible bounded symmetric domain in $\mathbb{C}^d$ of genus g in its Harish-Chandra realization. Let

$$\left\{1/\sqrt{V(\Omega)}, h_1(z), h_2(z), \cdots\right\}$$

be an orthonormal basis of the Hilbert space $A^2(\Omega)$ of square-integrable holomorphic functions on Ω. Define the Bergman kernel $K_\Omega(z, \bar{\xi})$ of Ω by

$$K_\Omega(z, \bar{\xi}) := 1/V(\Omega) + \sum_{i=1}^{\infty} h_i(z)\overline{h_i(\xi)}$$

for all $z, \xi \in \Omega$. Obviously, $1 \le V(\Omega)K_\Omega(z, \bar{z}) < +\infty$. The generic norm of Ω is defined by

$$N_\Omega(z, \bar{\xi}) := \left(V(\Omega)K_\Omega(z, \bar{\xi})\right)^{-\frac{1}{g}} \quad (z, \xi \in \Omega).$$

Thus $0 < N_\Omega(z, \bar{z}) \le 1$ for all $z \in \Omega$ and $N_\Omega(z, \bar{z}) = 0$ on the boundary of Ω.

For an irreducible bounded symmetric domain $\Omega \subset \mathbb{C}^d$ in its Harish-Chandra realization, a positive integer r and $\mathbf{n} = (n_1, \cdots, n_r) \in \mathbb{N}^r$, $\mathbf{p} = (p_1, \cdots, p_r) \in (\mathbb{R}_+)^r$, the Hua domain $H_\Omega(\mathbf{n}; \mathbf{p})$ is defined by

$$\begin{aligned} &H_\Omega(\mathbf{n}; \mathbf{p}) = H_\Omega(n_1, \cdots, n_r; p_1, \cdots, p_r) \\ &:= \left\{ (z, w_{(1)}, \cdots, w_{(r)}) \in \Omega \times \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_r} : \sum_{j=1}^{r} \|w_{(j)}\|^{2p_j} < N_\Omega(z, \bar{z}) \right\}, \end{aligned}$$

where $\| \cdot \|$ is the standard Hermitian norm. Note that $\Omega \times \{0\} \subset H_\Omega(\mathbf{n}; \mathbf{p})$ and $b\Omega \times \{0\} \subset bH_\Omega(\mathbf{n}; \mathbf{p})$ (where bD denotes the boundary of a domain D). Let the family $\Gamma(H_\Omega(\mathbf{n}; \mathbf{p}))$ be exactly the set of all mappings Φ:

$$\begin{aligned} &\Phi(z, w_{(1)}, \cdots, w_{(r)}) \\ &:= \left(\varphi(z),\ U_1(w_{(1)}) \frac{(N_\Omega(z_0, \overline{z_0}))^{\frac{1}{2p_1}}}{(N_\Omega(z, \overline{z_0}))^{\frac{1}{p_1}}},\ \cdots,\ U_r(w_{(r)}) \frac{(N_\Omega(z_0, \overline{z_0}))^{\frac{1}{2p_r}}}{(N_\Omega(z, \overline{z_0}))^{\frac{1}{p_r}}} \right) \end{aligned} \tag{5}$$

for $(z, w_{(1)}, \cdots, w_{(r)}) \in H_\Omega(\mathbf{n}; \mathbf{p})$, where $\varphi \in \mathrm{Aut}(\Omega)$, U_j is a unitary transformation of $\mathbb{C}^{n_j}$ for $1 \le j \le r$, and $z_0 = \varphi^{-1}(0)$. Then $\Gamma(H_\Omega(\mathbf{n}; \mathbf{p}))$ is a subgroup of the holomorphic automorphism group $\mathrm{Aut}(H_\Omega(\mathbf{n};\mathbf{p}))$ of $H_\Omega(\mathbf{n};\mathbf{p})$ (see Yin-Wang-Zhao-Zhao-Guan [YWZ]). Obviously, every element of $\Gamma(H_\Omega(\mathbf{n};\mathbf{p}))$ preserves the set $\Omega\times\{0\}(\subset H_\Omega(\mathbf{n};\mathbf{p}))$ and $\Gamma(H_\Omega(\mathbf{n};\mathbf{p}))$ is transitive on $\Omega\times\{0\}(\subset H_\Omega(\mathbf{n};\mathbf{p}))$. For the general reference of Hua domains, see Yin-Wang-Zhao-Zhao-Guan [YWZ] and references therein. When $r=1$, the Hua domain $H_\Omega(n_1;p_1)$ is also called the Cartan-Hartogs domain and is also denoted by $\Omega^{B^{n_1}}(p_1)$. For the reference of the Cartan-Hartogs domains, see Ahn-Byun-Park [ABP], Feng-Tu [FT1], Loi-Zedda [LZ] and Wang-Yin-Zhang-Roos [WYZ] and the references therein.

In 2012, Ahn-Byun-Park [ABP] determined the automorphism group of the Cartan-Hartogs domain $H_\Omega(n_1;p_1)$ by case-by-case checking only for four types of classical domains Ω. In 2014, by using a different technique from that in Ahn-Byun-Park [ABP], Tu-Wang [TW2] obtained the following result.

Theorem 4.3 (Tu-Wang) *Suppose that*

$$f: H_{\Omega_1}(\mathbf{n};\mathbf{p})\to H_{\Omega_2}(\mathbf{m};\mathbf{q})$$

is a biholomorphism between two equidimensional Hua domains $H_{\Omega_1}(\mathbf{n};\mathbf{p})$ and $H_{\Omega_2}(\mathbf{m};\mathbf{q})$ in their standard forms, where $\Omega_1\subset\mathbb{C}^{d_1}$ and $\Omega_2\subset\mathbb{C}^{d_2}$ are two irreducible bounded symmetric domains in the Harish-Chandra realization, and $\mathbf{n},\mathbf{m}\in\mathbb{N}^r$, $\mathbf{p},\mathbf{q}\in(\mathbb{R}_+)^r$. Then there exists an automorphism $\Phi\in\Gamma(H_{\Omega_2}(\mathbf{m};\mathbf{q}))$ (see (5) here) and a permutation $\sigma\in S_r$ with $n_{\sigma(i)}=m_i, p_{\sigma(i)}=q_i$ for $1\le i\le r$ such that

$$\Phi\circ f(z,w_{(1)},\cdots,w_{(r)})=(z,w_{(\sigma(1))},\cdots,w_{(\sigma(r))})\begin{pmatrix}A&&&\\&U_1&&\\&&\ddots&\\&&&U_r\end{pmatrix},$$

where A is a complex linear isomorphism of $\mathbb{C}^d$ $(d:=d_1=d_2)$ with $A(\Omega_1)=\Omega_2$, and U_i is a unitary transformation of $\mathbb{C}^{m_i}$ $(m_i=n_{\sigma(i)})$ for $1\le i\le r$.

As a special case of the above theorem, Tu-Wang [TW2] completely described the automorphism group of the Hua domains $H_\Omega(\mathbf{n};\mathbf{p})$ for all irreducible bounded symmetric domains Ω as follows.

Corollary 4.4 (Tu-Wang) *Let $H_\Omega(\mathbf{n};\mathbf{p})$ be a Hua domain in its standard form and $\Gamma(H_\Omega(\mathbf{n};\mathbf{p}))$ is generated by the mappings of the form (5), where $\Omega\subset\mathbb{C}^d$ is an irreducible bounded symmetric domain in the Harish-Chandra realization, and $\mathbf{n}\in\mathbb{N}^r$, $\mathbf{p}\in(\mathbb{R}_+)^r$. Then, for every $f\in\mathrm{Aut}(H_\Omega(\mathbf{n};\mathbf{p}))$, there exist a $\Phi\in\Gamma(H_\Omega(\mathbf{n};\mathbf{p}))$ and a permutation $\sigma\in S_r$ with $n_{\sigma(i)}=n_i$, $p_{\sigma(i)}=p_i$ for $1\le i\le r$ such that*

$$f(z,w_{(1)},\cdots,w_{(r)})=\Phi(z,w_{(\sigma(1))},\cdots,w_{(\sigma(r))}).$$

Also, there are many results (e.g., Dini-Primicerio [DP1, DP2], Hamada [Ha] and Landucci [L]) concerning proper holomorphic mappings between two generalized complex ellipsoids. For the case of $\mathbf{p},\mathbf{q}\in(\mathbb{Z}_+)^r$, in 1997, Dini-Primicerio ([DP2], Th. 4.6) proved the following result.

Theorem 4.5 (Dini-Primicerio) *Let* $\Sigma(\mathbf{n};\mathbf{p})$ *and* $\Sigma(\mathbf{m};\mathbf{q})$ *be two equidimensional generalized complex ellipsoids with* $\mathbf{n},\mathbf{m}\in\mathbb{N}^r$ *and* $\mathbf{p},\mathbf{q}\in(\mathbb{Z}_+)^r$ *(where* $p_k\neq 1,\ q_k\neq 1$ *for* $2\le k\le r$*) such that* $n_k\ge 2$ *whenever* $p_k\ge 2$ *and* $m_k\ge 2$ *whenever* $q_k\ge 2$ *for* $1\le k\le r$*. Then there exists a proper holomorphic mapping* $f:\Sigma(\mathbf{n};\mathbf{p})\to\Sigma(\mathbf{m};\mathbf{q})$ *if and only if there exists a permutation* $\sigma\in S_r$ *such that* $n_{\sigma(j)}=m_j,\ p_{\sigma(j)}=q_j$ *for* $1\le j\le r$*.*

Remark on Theorem 4.5. When $\mathbf{p},\mathbf{q}\in(\mathbb{Z}_+)^r$, $\Sigma(\mathbf{n};\mathbf{p})$ and $\Sigma(\mathbf{m};\mathbf{q})$ are pseudoconvex domains with real analytic boundaries. The conditions "$n_i\ge 2$ whenever $p_i\ge 2$ and $m_i\ge 2$ whenever $q_i\ge 2$ for $1\le i\le r$" are indispensable in proving this theorem.

For the Hua domain $H_\Omega(\mathbf{n};\mathbf{p})=H_\Omega(n_1,\cdots,n_r;p_1,\cdots,p_r)$ in its standard form, the boundary $bH_\Omega(\mathbf{n};\mathbf{p})$ of $H_\Omega(\mathbf{n};\mathbf{p})$ is comprised of

$$bH_\Omega(\mathbf{n};\mathbf{p})=b_0H_\Omega(\mathbf{n};\mathbf{p})\cup b_1H_\Omega(\mathbf{n};\mathbf{p})\cup(b\Omega\times\{0\}), \tag{6}$$

where

$$b_0H_\Omega(\mathbf{n};\mathbf{p}):=\Big\{(z,w_{(1)},\cdots,w_{(r)})\in\Omega\times\mathbb{C}^{n_1}\times\cdots\times\mathbb{C}^{n_r}:\ \sum_{i=1}^r\|w_{(i)}\|^{2p_i}=N_\Omega(z,z),\|w_{(j)}\|^2\neq 0,1+\delta\le j\le r\Big\},$$

$$b_1H_\Omega(\mathbf{n};\mathbf{p}):=\bigcup_{j=1+\delta}^{r}\Big\{(z,w_{(1)},\cdots,w_{(r)})\in\Omega\times\mathbb{C}^{n_1}\times\cdots\times\mathbb{C}^{n_r}:\ \sum_{i=1}^r\|w_{(i)}\|^{2p_i}=N_\Omega(z,z),\|w_{(j)}\|^2=0\Big\},$$

in which

$$\delta=\begin{cases}1 & \text{if } p_1=1,\\ 0 & \text{if } p_1\neq 1.\end{cases}$$

Then, (a) $b_0H_\Omega(\mathbf{n};\mathbf{p})$ is a real analytic hypersurface in $\mathbb{C}^{d+|\mathbf{n}|}$ and $H_\Omega(\mathbf{n};\mathbf{p})$ is strongly pseudoconvex at all points of $b_0H_\Omega(\mathbf{n};\mathbf{p})$; (b) If $H_\Omega(\mathbf{n};\mathbf{p})$ isn't a ball, then $H_\Omega(\mathbf{n};\mathbf{p})$ is not strongly pseudoconvex at any point of $b_1H_\Omega(\mathbf{n};\mathbf{p})\cup(b\Omega\times\{0\})$. Obviously, $b_1H_\Omega(\mathbf{n};\mathbf{p})\cup(b\Omega\times\{0\})$ is contained in a complex analytic subset in $\mathbb{C}^{d+|\mathbf{n}|}$ of complex codimension equal to $\min\{n_{1+\delta},\cdots,n_r,n_1+\cdots+n_r\}$ (note $\min\{n_{1+\delta},\cdots,n_r,n_1+\cdots+n_r\}=n_1$ for $r=1$ and $\min\{n_{1+\delta},\cdots,n_r,n_1+\cdots+n_r\}=\min\{n_{1+\delta},\cdots,n_r\}$ for $r\ge 2$). Tu-Wang [TW2] proved the rigidity of proper holomorphic mappings between two equidimensional Hua domains as follows.

Theorem 4.6 (Tu-Wang) *Suppose that*

$$f:H_{\Omega_1}(\mathbf{n}_1;\mathbf{p}_1)\to H_{\Omega_2}(\mathbf{n}_2;\mathbf{p}_2)$$

is a proper holomorphic mapping between two equidimensional Hua domains $H_{\Omega_1}(\mathbf{n}_1;\mathbf{p}_1)$ *and* $H_{\Omega_2}(\mathbf{n}_2;\mathbf{p}_2)$ *in their standard forms, where* $\Omega_1\subset\mathbb{C}^{d_1}$ *and* $\Omega_2\subset\mathbb{C}^{d_2}$ *are two irreducible bounded symmetric domains in the Harish-Chandra realization, and*

$\mathbf{n}_1, \mathbf{n}_2 \in \mathbb{N}^r$, $\mathbf{p}_1, \mathbf{p}_2 \in (\mathbb{R}_+)^r$. *Assume that* $b_1 H_{\Omega_i}(\mathbf{n}_i; \mathbf{p}_i) \cup (b\Omega_i \times \{0\})$ $(i = 1, 2)$ *is contained in some complex analytic set of complex codimension at least* 2. *Then* $f : H_{\Omega_1}(\mathbf{n}_1; \mathbf{p}_1) \to H_{\Omega_2}(\mathbf{n}_2; \mathbf{p}_2)$ *is a biholomorphism.*

Remarks on Theorem 4.6. (i) In Theorem 4.6, we do not assume $\dim \Omega_1 = \dim \Omega_2$. (ii) In Theorem 4.6, the assumption "$b_1 H_\Omega(\mathbf{n}; \mathbf{p}) \cup (b\Omega \times \{0\})$ is contained in some complex analytic set of complex codimension at least 2" is equivalent to that $H_\Omega(\mathbf{n}; \mathbf{p})$ (in its standard form) satisfies

$$\min\{n_{1+\delta}, \cdots, n_r, n_1 + \cdots + n_r\} \geq 2,$$

that is, $H_\Omega(\mathbf{n}; \mathbf{p})$ (in its standard form) satisfies the following assumptions: (a) If $\Omega = \mathbf{B}_d$ is the unit ball, then $\min\{n_1, \cdots, n_r\} \geq 2$; (b) If $\text{rank}(\Omega) \geq 2$ and $p_1 \neq 1$, then $\min\{n_1, \cdots, n_r\} \geq 2$; (c) If $\text{rank}(\Omega) \geq 2$ and $p_1 = 1$, then $\min\{n_2, \cdots, n_r, n_1 + n_2 + \cdots + n_r\} \geq 2$. (iii) In Theorem 4.6, the assumption " $b_1 H_{\Omega_i}(\mathbf{n}_i; \mathbf{p}_i) \cup (b\Omega_i \times \{0\})$ $(i = 1, 2)$ is contained in some complex analytic set of complex codimension at least 2" cannot be removed.

Combining the above results, Tu-Wang [TW2] immediately obtained the following result.

Corollary 4.7 (Tu-Wang) *Suppose that* f *is a proper holomorphic self-mapping on the Hua domain* $H_\Omega(\mathbf{n}; \mathbf{p})$ *in its standard form, where* $\Omega \subset \mathbb{C}^d$ *is an irreducible bounded symmetric domain in the Harish-Chandra realization, and* $\mathbf{n} \in \mathbb{N}^r$, $\mathbf{p} \in (\mathbb{R}_+)^r$ *with* $\min\{n_{1+\delta}, \cdots, n_r, n_1 + n_2 + \cdots + n_r\} \geq 2$. *Then* f *is an automorphism of the Hua domain* $H_\Omega(\mathbf{n}; \mathbf{p})$, *that is, there exist a* $\Phi \in \Gamma(H_\Omega(\mathbf{n}; \mathbf{p}))$ *and a permutation* $\sigma \in S_r$ *with* $n_{\sigma(i)} = n_i$, $p_{\sigma(i)} = p_i$ *for* $1 \leq i \leq r$ *such that*

$$f(z, w_{(1)}, \cdots, w_{(r)}) = \Phi(z, w_{(\sigma(1))}, \cdots, w_{(\sigma(r))}).$$

For other related research, we refer the reader to Ahn-Park [AP], Feng-Tu [FT2], Kim-Ninh-Yamamori [KNY], Kosiński [Ko], Su-Tu-Wang [STW], Tu-Wang [TW1], Yin [Yin2] and the references therein.

References

[ABP] H. Ahn, J. Byun and J. Park, Automorphisms of the Hartogs type domains over classical symmetric domains. *Internat. J. Math.* **23**(9): 1250098, (2012), 11 pp

[AP] H. Ahn and J. Park, The explicit forms and zeros of the Bergman kernel function for Hartogs type domains. *Journal of Functional Analysis* **262**(8), 3518-3547 (2012)

[Alx] H. Alexander, Proper holomorphic mappings in $\mathbf{C}^n$. *Indiana Univ. Math. Journal* **26**, 137–146(1977)

[AHJY] J. Andrews, X. Huang, S. Ji and W. Yin, Mapping from $\mathbb{B}^n$ to $\mathbb{B}^{3n-3}$, to appear in *Communications in Analysis and Geometry.*

[BH] M.S. Baouendi and X. Huang, Super-rigidity for holomorphic mappings between hyperquadrics with positive signature. *J. Diff. Geom.* **69**(2), 379–398 (2005)

[BEH] M.S. Baouendi, P. Ebenfelt and X. Huang, Holomorphic mappings between hyperquadrics with small signature difference. *Amer. J. Math.* **133**(6), 1633–1661 (2011)

[CMk] H. Cao and N. Mok, Holomorphic immersions between compact hyperbolic space forms. *Invent. Math.* **100**, 49–61 (1990)

[CM] S.S. Chern and J.K. Moser, Real hypersurfaces in complex manifolds. *Acta Math.* **133**, 219–271 (1974)

[CS1] J. Cima and T.J. Suffridge, A reflection principle with applications to proper holomorphic mappings. *Math. Ann.* **265**, 489–500 (1983)

[DA] J.P. D'Angelo, *Several Complex Variables and the Geometry of Real Hypersurfaces*, (CRC Press, Boca Raton, 1993)

[DL] J.P. D'Angelo and J. Lebl, Complexity results for CR mappings between spheres. *Internat. J. Math.* **20**(2), 149–166 (2009)

[DP1] G. Dini and A. Primicerio, Proper holomorphic mappings between generalized pseudoellipsoids. *Annali di matematica pura ed applicata* **158**, 219–229 (1991)

[DP2] G. Dini and A. Primicerio, Localization principle of automorphisms on generalized pseudoellipsoids. *J. Geom. Anal.* **7**(4), 575–584 (1997)

[Eb] P. Ebenfelt, Partial rigidity of degenerate CR embeddings into spheres. *Advances in Mathematics* **239**, 72–96 (2013)

[Fa1] J. Faran, Maps from the two ball to the three ball. *Invent. Math.* **68**, 441–475 (1982)

[Fa2] J. Faran, On the linearity of proper holomorphic maps between balls in the low codimensional case. *J. Diff. Geom.* **24**, 15–17 (1986)

[FT1] Z. Feng and Z. Tu, On canonical metrics on Cartan-Hartogs domains. *Math. Z.* **278**, 301–320 (2014)

[FT2] Z. Feng and Z. Tu, Balanced metrics on some Hartogs type domains over bounded symmetric domains. *Annals of Global Analysis and Geometry* **47**, 305–333 (2015)

[Fo1] F. Forstneric, Extending proper holomorphic mappings of positive codimension. *Invent. Math.* **95**(1), 31–62 (1989)

[Fo2] F. Forstneric, A survey on proper holomorphic mappings, Proceeding of Year in SCVs at Mittag-Leffler Institute, Math. Notes **38** (Princeton University Press, Princeton, N.J., 1992)

[GN] Y. Gao and S.-C. Ng, On the classification of proper holomorphic mappings among generalized complex balls, *In preparation.*

[HS] M. Hakim and N. Sibony, Fonctions holomorphes bornées sur la boule unité de $\mathbf{C}^n$. *Invent. Math.* **67**(2), 213–222 (1982)

[Ha] H. Hamada, On proper holomorphic self-maps of generalized complex ellipsoids. *J. Geom. Anal.* **8**, 441–446 (1998)

[Ha1] H. Hamada, Rational proper holomorphic maps from $\mathbb{B}^n$ into $\mathbb{B}^{2n}$. *Math. Ann.* **331**(3), 693–711 (2005)

[HN] G.M. Henkin and R. Novikov, Proper mappings of classical domains, in *Linear and Complex Analysis Problem Book*, Lecture Notes in Math. Vol. **1043** (Springer, Berlin, 1984), pp.625–627

[HT] G.M. Henkin and A.E. Tumanov, Local characterization of analytic automorphisms of classical domains (Russian), *Dokl. Akad. Nauk SSSR* **267**, 796-799(1982); English translation: *Math. Notes* **32**, 849–852(1982)

[Hu1] X. Huang, On a linearity problem for proper holomorphic maps between balls in complex spaces of different dimensions. *J. Diff. Geom.* **51**, 13–33 (1999)

[Hu2] X. Huang, On a semi-rigidity property for holomorphic maps. *Asian Jour. of Math.***7**(4), 463–492 (2003)

[HJ] X. Huang and S. Ji, Mapping $\mathbb{B}^n$ into $\mathbb{B}^{2n-1}$. *Invent. Math.* **145**(2), 219–250 (2001)

[HJX1] X. Huang, S. Ji and D. Xu, A new gap phenomenon for proper holomorphic mappings from $\mathbb{B}^n$ to $\mathbb{B}^N$. *Math Research Letter* **3**(4), 515–529 (2006)

[HJX2] X. Huang, S. Ji and D. Xu, Several results for holomorphic mappings from $\mathbb{B}^n$ into $\mathbb{B}^N$, Geometric analysis of PDE and several complex variables, Contemp. Math. **368** (Amer. Math. Soc., Providence, RI, 2005), pp.267–292

[HJY1] X. Huang, S. Ji and W. Yin, Recent progress on two problems in several complex variables, *Proceedings of International Congress of Chinese Mathematicians 2007*, Vol. I (International Press, 2009), 563–575

[HJY] X. Huang, S. Ji and W. Yin, On the third gap for proper holomorphic maps between Balls. *Math. Ann.* **358** (1-2), 115–142 (2014)

[Hw] J.-M. Hwang, Geometry of varieties of minimal rational tangents, *Current Developments in Algebraic Geometry.* MSRI Publications, Volume **59**, 197–226(2011)

[HY1] X. Huang and W. Yin, A Bishop surface with a vanishing Bishop invariant. *Inventiones Mathematicae* **176**(3), 461–520 (2009)

[HY2] X. Huang and W. Yin, A codimension two CR singular submanifold that is formally equivalent to a symmetric quadric. *International Mathematics Research Notices* **15**, 2789–2828 (2009)

[HY3] X. Huang and W. Yin, Flattening of CR singular points and analyticity of local hull of holomorphy I, to appear in *Math. Ann.*

[HY4] X. Huang and W. Yin, Flattening of CR singular points and analyticity of local hull of holomorphy II, to appear in *Adv. Math.*

[KNY] H. Kim, V.T. Ninh and A. Yamamori, The automorphism group a certain unbounded non-hyperbolic domain. *J. Math. Anal. Appl.* **409**, 637–642 (2014)

[KZ] S.-Y. Kim and D. Zaitsev, Rigidity of proper holomorphic maps between bounded symmetric domains. *Math. Ann.*, DOI 10.1007/s00208-014-1143-2.

[Kod] A. Kodama, On the holomorphic automorphism group of a generalized complex ellipsoid. *Complex Variables and Elliptic Equations* **59**, 1342–1349 (2014)

[KKM] A. Kodama, S.G. Krantz and D. Ma, A characterization of generalized complex ellipsoids in $\mathbb{C}^N$ and related results. *Indiana Univ. Math. J.* **41**, 173–195 (1992)

[Ko] L. Kosiński, The group of automorphisms of the pentablock. *Complex Anal. Oper. Theory*, DOI: 10.1007/s11785-014-0422-y, (2015), 12pp.

[L] M. Landucci, On the proper holomorphic equivalence for a class of pseudoconvex domains. *Trans. Amer. Math. Soc.* **282**, 807–811 (1984)

[LZ] A. Loi and M. Zedda, Balanced metrics on Cartan and Cartan-Hartogs domains. *Math. Z.* **270**, 1077-1087(2012)

[Low] E. Low, Embeddings and proper holomorphic maps of strictly pseudoconvex domains into polydiscs and balls. *Math. Z.* **190**(3), 401–410 (1985)

[Mok] N. Mok, Metric Rigidity Theorems on Hermitian Locally Symmetric Manifolds. Series in Pure Mathematics, Vol. 6, (World Scientific, Singapore, 1989)

[Mok1] N. Mok, Uniqueness theorems of Hermitian metric of seminegative curvature on quotients of bounded symmetric domains. *Ann. Math.* **125**, 105–152 (1987)

[Mok2] N. Mok, Uniqueness theorem of Kähler metrics of semipositive holomorphic bisectional curvature on compact Hermitian symmetric space. *Math. Ann.* **276**, 177–204 (1987)

[Mok3] N. Mok, Geometric structures on uniruled projective manifolds defined by their varieties of minimal rational tangents. *Géométrie différentielle, physique mathématique, mathématiques et société*, Vol.2, Astérisque **322**, (Soc. Math. de France, Paris, 2008), pp.151–205

[Mok4] N. Mok, Nonexistence of proper holomorphic maps between certain classical bounded symmetric domains. *Chi. Ann. Math. Ser. B.* **29**(2), 135–146(2008)

[MN] N. Mok and S.-C. Ng, Germs of measure-preserving maps from bounded symmetric domains to their Cartesian products. *J. Reine Angew. Math.* **669**, 47–72(2012)

[MNT] N. Mok, S.-C. Ng and Z. Tu, Factorization of proper holomorphic maps on irreducible bounded symmetric domains of rank ≥ 2. *Sci. China Math.* **53**(3), 813–826 (2010)

[MT] N. Mok and I.-H. Tsai, Rigidity of convex realizations of irreducible bounded symmetric domains of rank ≥ 2. *J. Reine Angew. Math.* **431**, 91–122(1992)

[Mos] G. Mostow, Strong Rigidity of Locally Symmetric Spaces. Annals of Math Studies, vol. 78 (Princeton University Press, Princeton, NJ, 1973)

[Nar] I. Naruki, The holomorphic equivalence problem for a class of Reinhardt domains. *Publ. Res. Inst. Math. Sci., Kyoto Univ.* **4**, 527–543 (1968)

[Ng] S.-C. Ng, On proper holomorphic mappings among irreducible bounded symmetric domains of rank at least 2. *Proc. Amer. Math. Soc.* **143**, 219–225(2015)

[Ng1] S.-C. Ng, Holomorphic Double Fibration and the Mapping Problems of Classical Domains. *Int. Math. Res. Not.* **2015**(2), 291–324(2015)

[Ng2] S.-C. Ng, Cycle spaces of flag domains on Grassmannians and rigidity of holomorphic mappings. *Math. Res. Lett.* **19**(6), 1219–1236 (2012)

[Ng3] S.-C. Ng, Proper holomorphic mappings on flag domains of SU(p,q)-type on projective spaces. *Michigan. Math. J.* **62**(4), 769–777 (2013)

[Po] H. Poincaré, Les fonctions analytiques de deux variables et la représentation conforme. *Ren. Cire. Mat. Palermo II.* **23**, 185–220 (1907)

[Re1] R. Remmert, Projectionen analytischer Mengen. *Math. Ann.* **130**, 410–441 (1956)

[Re2] R. Remmert, Holomorphe und meromorphe Abbildungen komplexer Räume. *Math. Ann.* **133**, 328–370 (1957)

[Seo] A. Seo, New examples of proper holomorphic maps among symmetric domains. *To appear in Michigan. Math. J.*

[Siu1] Y.T. Siu, The complex analyticity of harmonic maps and the strong rigidity of compact Kähler manifolds. *Ann. Math.* **112**, 73–111 (1980)

[Siu2] Y.T. Siu, Strong rigidity of compact quotients of exceptional bounded symmetric domains. *Duke Math. J.* **48**, 857–871 (1981)

[Siu3] Y.T. Siu, Some recent results in complex manifold theory related to vanishing theorems for the semipositive case. Lecture Notes in Math. Vol. **1111** (Springer, Berlin, 1985), pp.169–192

[Ste] B. Stensones, Proper maps which are Lipschitz α up to the boundary. *J. Geom. Analysis* **6**(2), 317–339 (1996)

[STW] G. Su, Z. Tu and L. Wang, Rigidity of proper holomorphic self-mappings of the pentablock. *J. Math. Anal. Appl.* **424**, 460–469 (2015)

[Tsai] I-H. Tsai, Rigidity of proper holomorphic maps between symmetric domains. *J. Diff. Geom.* **37**, 123–160 (1993)

[Tu] Z. Tu, Rigidity of proper holomorphic mappings between equidimensional bounded symmetric domains. *Proc. Amer. Math. Soc.* **130**(4), 1035–1042 (2001)

[Tu1] Z. Tu, Rigidity of proper holomorphic mappings between nonequidimensional bounded symmetric domains. *Math. Z.* **240**, 13–35 (2002)

[TW1] Z. Tu and L. Wang, Rigidity of proper holomorphic mappings between certain unbounded non-hyperbolic domains. *J. Math. Anal. Appl.* **419**, 703–714 (2014)

[TW2] Z. Tu and L. Wang, Rigidity of proper holomorphic mappings between equidimensional Hua domains. *Math. Ann.* 2015, DOI: 10.1007/s00208-014-1136-1, 34pp, in press.

[WYZ] A. Wang, W.P. Yin, L.Y. Zhang and G. Roos, The Kähler-Einstein metric for some Hartogs domains over bounded symmetric domains. *Science China Series A: Math.* **49**(9), 1175–1210 (2006)

[W] S. Webster, On mapping an n-ball into an (n+1)-ball in the complex space. *Pacific Journal Math.* **81**, 267–272 (1979)

[Wo] J.A. Wolf, Fine structure of Hermitian symmetric spaces. *Geometry of Symmetric Spaces*, ed. Boothby-Weiss(Marcel-Dekker, New York, 1972), pp.271–357

[Yin1] W.K. Yin, Divergent Birkhoff normal forms of real analytic area preserving maps. *Math. Z.* DOI: 10.1007/s00209-015-1460-5, in press.

[Yin2] W.P. Yin, The Summarizations on Research of Hua Domain. *Advances in Math.* (China) **36**(2), 129–152(2007)

[YWZ] W.P. Yin, A. Wang, Z. Zhao, X. Zhao and B. Guan, The Bergman kernel functions on Hua domains. *Science in China Series A: Math.* **44**(6), 727–741(2001)